China’s Model of Managing the Financial System

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Online Appendix

This online appendix present proofs of the propositions in the main paper.

Proof of Proposition 1

Note from the variance of the excess asset payoff that:

$$Var[R_{t+1} | \mathcal{F}_t] = \sigma_D^2 + \left( \frac{1}{R^f - \rho_V} \right)^2 \sigma_V^2 + p_N^2 \sigma_N^2,$$

and thus the excess volatility is driven by the $p_N^2 \sigma_N^2$ term. Consider now the expression for the less positive root of $p_N$ from Proposition 3 in the special case in which there is an absence of government intervention:

$$p_N = \frac{1}{2 \sigma_N^2} A - \sqrt{\left( \frac{1}{2 \sigma_N^2} A \right)^2 - \frac{1}{\sigma_N^2} B},$$

where:

$$A = \frac{R^f}{\gamma},$$

$$B = \sigma_D^2 + \left( \frac{\sigma_V}{R^f - \rho_V} \right)^2,$$

to simplify notation. Given this expression, it follows that:

$$p_N^2 \sigma_N^2 = \frac{A^2}{2 \sigma_N^2} - A \sqrt{\left( \frac{A}{2 \sigma_N^2} \right)^2 - \frac{1}{\sigma_N^2} B} - B.$$

Differentiating with respect to $\sigma_N^2$, we find with some manipulation that:

$$\frac{\partial p_N^2 \sigma_N^2}{\partial \sigma_N^2} = \frac{A}{2 \sigma_N^2} 2 \left( \frac{A}{2 \sigma_N^2} \right)^2 - \frac{1}{\sigma_N^2} B - \frac{A}{\sigma_N^2} \sqrt{\left( \frac{A}{2 \sigma_N^2} \right)^2 - \frac{1}{\sigma_N^2} B},$$

$$\sqrt{\left( \frac{A}{2 \sigma_N^2} \right)^2 - \frac{1}{\sigma_N^2} B},$$
which we can factorize as:

$$\frac{\partial \rho_N^2 \sigma_N^2}{\partial \sigma_N^2} = \frac{A}{2\sigma_N^2} \left( \frac{A}{2\sigma_N} - \sqrt{\left( \frac{A}{2\sigma_N} \right)^2 - \frac{1}{\sigma_N} B} \right)^2 \geq 0,$$

and, since \( P_t = \frac{1}{R^f - \rho_V} V_{t+1} + p_N N_t \) with \( V_{t+1} \) and \( N_t \) independent of each other, this completes the proof. Therefore, volatility is highest close to market breakdown, when \( \left( \frac{R_f}{\sqrt{\sigma_N^2}} \right)^2 - 4 \left( \frac{\sigma_D^2}{\sigma_N^2} + \left( \frac{\sigma_V}{R^f - \rho_V} \right)^2 \right) = \varepsilon \) for \( \varepsilon \) arbitrarily small. Market breakdown occurs when \( \varepsilon = 0 \), or:

$$\sigma_N = \frac{R_f}{2\gamma \left( \frac{\sigma_D^2}{\sigma_N^2} + \left( \frac{\sigma_V}{R^f - \rho_V} \right)^2 \right)}.$$

Furthermore, as \( \varepsilon \to 0 \), and \( \sigma_N \to \frac{R_f}{2\gamma \left( \frac{\sigma_D^2}{\sigma_N^2} + \left( \frac{\sigma_V}{R^f - \rho_V} \right)^2 \right)} \), then:

$$p_N^2 \sigma_N^2 \to \sigma_D^2 + \left( \frac{\sigma_V}{R^f - \rho_V} \right)^2.$$

Consequently, the maximum conditional excess payoff variance before breakdown occurs is

$$\text{Var} \left[ R_{t+1} \mid \mathcal{F}_t \right] \to 2 \left( \sigma_D^2 + \left( \frac{\sigma_V}{R^f - \rho_V} \right)^2 \right).$$

### Proof of Proposition 2

In what follows, let \( \tau \) be the maximum horizon (remaining trading periods) of all investors and \( T \leq \tau \) be the horizon of an individual investor. We first solve the optimization problem of an individual investor whose current investment horizon is \( T \). We then impose market-clearing, in which we aggregate the demands of \( \tau \) vintages of investors to solve for asset prices. Finally, we consider the two cases where \( \tau = 2 \) and \( \tau = \infty \).

We search for a covariance-stationary equilibrium, and conjecture a price process:

$$P_t = p_V V_{t+1} + p_N N_t$$

it will be convenient to define the state vector \( \Psi_t = [1, v_t, N_t] \), and we search for a covariance-stationary equilibrium. \( \Psi_t \) has an AR(1) law of motion and is related to investors’ returns, \( R_{t+1} \), according to:

$$\Psi_{t+1} = \phi \Psi_t + \Xi \varepsilon_{t+1},$$

$$R_{t+1} = \alpha \Psi_t + \varphi' \varepsilon_{t+1},$$
where $E \left[ R_{t+1} \mid \mathcal{F}_t \right] = \alpha \Psi_t$. In the limiting covariance-stationary equilibrium of the economy:

$$
\varphi = \begin{bmatrix}
1 & 0 & 0 \\
0 & \rho_N & 0 \\
0 & 0 & 0
\end{bmatrix},
$$

$$
\varphi' = \begin{bmatrix}
1 & \rho_N \\
0 & 0 \\
0 & 0
\end{bmatrix},
$$

$$
\Xi = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
$$

$$
\alpha = \begin{bmatrix}
0 & 1 + (\rho_N - R^f) p_N & -R^f p_N
\end{bmatrix}.
$$

As such, we can write the law of motion for an investor’s wealth with horizon $T$ at date $t$ as:

$$
W_{t+1}^{T-1} = R^f W_t^T + X_t^T \left( \alpha \Psi_t + \varphi' \varepsilon_{t+1} \right) - C_t^T.
$$

Let us now conjecture that the value function of an investor with horizon $T$ at date $t$, $V_t(T, W_t, \Psi_t)$, takes the functional form:

$$
V_t(T, W_t, \Psi_t) = -\exp \left( -A_t^T W_t^T - \frac{1}{2} \Psi_t' B_t^T \Psi_t \right).
$$

Substituting this expression into the recursive formulation of the investor’s problem, we find that:

$$
V_t = \sup_{\{C_t^T, X_t^T\}} -\exp \left( -\gamma C_t^T \right) - \beta E \left[ \exp \left( -A_{t+1}^T R^f (W_t^T - C_t^T) - A_{t+1}^T X_t^T \alpha \Psi_t - A_{t+1}^T X_t^T \varphi' \varepsilon_{t+1} \right) \right] \mid \mathcal{F}_t.
$$

By completing the square, we can rewrite the above Bellman Equation as:

$$
V_t^G = \sup_{\{C_t^T, X_t^T\}} -\exp \left( -\gamma C_t^T \right) - \frac{1}{\sqrt{\det(\Omega)}} \beta
$$

$$
\times \exp \left( -A_{t+1}^T R^f (W_t^T - C_t^T) - A_{t+1}^T X_t^T \alpha \Psi_t - A_{t+1}^T X_t^T \varphi' \varepsilon_{t+1} \right)
$$

Taking the first-order conditions with respect to $X_t^T$ and $C_t^T$, we find that the optimal position and consumption of the government take the form:

$$
X_t^T = \frac{\alpha - \varphi' (\Xi' B_{t+1}^T \Xi + \Omega^{-1})^{-1} \Xi' B_{t+1}^{T-1} \theta \Psi_t}{A_{t+1}^T \varphi' (\Xi' B_{t+1}^T \Xi + \Omega^{-1})^{-1} \varphi},
$$

and:

$$
C_t^T = \frac{A_{t+1}^T R^f}{\gamma + A_{t+1}^T R^f} W_t^T + \frac{1}{\gamma + A_{t+1}^T R^f} \log \left( \frac{\gamma}{A_{t+1}^T \beta R^f} \sqrt{\det(\Xi' B_{t+1}^T \Xi + \Omega^{-1})} \right)
$$

$$
+ \frac{1}{2 (\gamma + A_{t+1}^T R^f)} \Psi_t' B_{t+1}^{T-1} \theta \Psi_t + \frac{A_{t+1}^T}{\gamma + A_{t+1}^T R^f} X_t^T \alpha \Psi_t
$$

$$
- \frac{1}{2 (\gamma + A_{t+1}^T R^f)} (A_{t+1}^T X_t^T \varphi + \Psi_t' B_{t+1}^{T-1} \Xi) (\Xi' B_{t+1}^{T-1} \Xi + \Omega^{-1})^{-1} (A_{t+1}^T X_t^T \varphi + \Xi' B_{t+1}^{T-1} \theta \Psi_t).
$$
Substituting for $X^T_t$, optimal consumption takes the form:

$$C^T_t = \frac{A^{T-1}_t R^f}{\gamma + A^{T-1}_t R^f} W^T_t + \tilde{C}^{T-1}_{t+1} + \frac{1}{2 (\gamma + A^{T-1}_t R^f)} \Psi_t A^{T-1}_t \Psi_t,$$

where:

$$\tilde{C}^{T-1}_{t+1} = \frac{1}{\gamma + A^{T-1}_t R^f} \log \left( \frac{\gamma}{\beta A^{T-1}_t R^f} \sqrt{|\Omega| |\Xi' B^{T-1}_{t+1} \Xi + \Omega^{-1}|} \right),$$

$$A^{T-1}_{t+1} = \varphi' \left( B^{T-1}_{t+1} - B^{T-1}_{t+1} \Xi (\Xi' B^{T-1}_{t+1} \Xi + \Omega^{-1})^{-1} \Xi' B^{T-1}_{t+1} \varphi \right) \varphi' (\Xi' B^{T-1}_{t+1} \Xi + \Omega^{-1})^{-1} \varphi.$$

Substituting the optimal policies into the maximized Bellman Equation, we find that:

$$\gamma C^T_t - A^T_t W^T_t - \frac{1}{2} \Psi_t B^T_t \Psi_t = \log \left( 1 + \frac{\gamma}{A^{T-1}_t R^f} \right),$$

from which follows that:

$$A^T_t = \frac{\gamma R^f - 1}{R^f - (R^f)^T},$$

$$\tilde{C}^{T-1}_{t+1} = \frac{A^T_t}{\gamma A^{T-1}_t R^f} \log \left( \frac{\gamma}{\beta A^{T-1}_t R^f} \sqrt{|\Omega| |\Xi' B^{T-1}_{t+1} \Xi + \Omega^{-1}|} \right),$$

with $A_T = \gamma$.

Furthermore, $B_t$ satisfies the finite difference system of equations:

$$\frac{1}{2} \Psi'_t \left( 2 \left( \gamma \tilde{C}^{T-1}_{t+1} + \log \left( \frac{R^f - 1}{R^f} \right) e^1_{3 \times 1} e^\nu_{3 \times 1} \right) + \frac{1}{R^f} A^{T-1}_{t+1} - B^T_t \right) \Psi_t = 0,$$

and since they must hold for arbitrary $\Psi_t$, it follows that $B^T_t$ is determined from $B^{T-1}_{t+1}$ by:

$$B^T_t = \frac{1}{R^f} A^{T-1}_{t+1} + 2 \left( \gamma \tilde{C}^{T-1}_{t+1} + \log \left( \frac{R^f - 1}{R^f} \right) \right) e^1_{3 \times 1} e^\nu_{3 \times 1},$$

where $e^1_{3 \times 1}$ is the $3 \times 1$ basis vector with first entry 1 and remaining entries 0. Recognizing that at the final date for a cohort;

$$B^0_t = 0_{3 \times 3},$$

it follows that one can iterate backward in time to arrive at $B^T_t$. In the limit of arbitrarily large $T$, one can find the stationary fixed point $B_t \to B$ since all variances and covariances governing the law of motion of $\Psi$ and the mapping to $R_{t+1}$ are stationary.
The optimal consumption and investment plans can be summarized as:

\[
(\xi'B_t^T\xi + \Omega^{-1})^{-1} = \begin{bmatrix}
\sigma_D^{-2} & 0 & B_{t,22}^T + \sigma_v^{-2} & B_{t,23}^T \\
0 & B_{t,32}^T & B_{t,33}^T & \sigma_N^{-2}
\end{bmatrix}^{-1} = \begin{bmatrix}
\sigma_D^2 & 0 & 0 & 0 \\
0 & -B_{t,33}^T \frac{\Delta_t}{\Delta_t} & -B_{t,23}^T \frac{\Delta_t}{\Delta_t} & 0 \\
0 & -B_{t,32}^T \frac{\Delta_t}{\Delta_t} & B_{t,22}^T + \sigma_v^{-2} \frac{\Delta_t}{\Delta_t} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

being positive definite, where:

\[
\Delta_t = (B_{t,22}^T + \sigma_v^{-2}) (B_{t,33}^T + \sigma_N^{-2}) - B_{t,23}^T B_{t,32}^T,
\]

which we will verify in the sequel that it is always satisfied.

This completes our characterization of the solution to dynamic optimization problem.

The optimal consumption and investment plans can be summarized as:

\[
C_t^T = \frac{R^t - 1}{R^t} W_t^T - \frac{1}{\gamma R^t} \log \left( \left( R^t - 1 \right) \beta \sqrt{\| \xi'B_t^T\xi + \Omega^{-1} \|} \right) + \frac{1}{2\gamma R^t} \Psi_t^T \Lambda_{t+1}^T \Psi_t,
\]

\[
X_t^T = \frac{R^t \left( 1 - (R^t)^{t-T} \right)}{R^t - 1} \alpha - \varphi' (\xi'B_t^T\xi + \Omega^{-1})^{-1} (\xi'B_t^T \varphi)^T \Psi_t.
\]

Given the equivalence of the sequential and dynamic problems by the Dynamic Programming Principle, the solution solves the investor’s problem.

Given our expression for \((\xi'B_t^T\xi + \Omega^{-1})^{-1}\), it follows that:

\[
X_t = \sum_{t'=t}^{T} \left( \frac{1}{R^t} \right)^{t'-1} \frac{1}{\gamma} \left[ (1 + (\rho_V - R^t) p_V) V_t - R^t p_N N_t - \frac{B_{t,33}^T + \sigma_N^{-2}}{\Delta_t} p_N - \frac{B_{t,23}^T + \sigma_v^{-2}}{\Delta_t} p_V \right] \\
\quad \times \left[ \frac{B_{t,12}^T + B_{t,22}^T}{\Delta_t} \rho_V V_t \right]
\]

Since there are overlapping generations of investors with horizons from 1 to \(\tau\), it follows that there is a stationary equilibrium for the asset price. Market-clearing and matching coefficients implies that

\[
-\frac{1}{T} \sum_{t=0}^{T} \left( 1 - (R^t)^{t-T} \right) \left( \frac{B_{t,33}^T + \sigma_N^{-2}}{\Delta_t} p_N - \frac{B_{t,32}^T + \sigma_v^{-2}}{\Delta_t} p_V \right) B_{t,12}^T + \left( -\frac{B_{t,22}^T + \sigma_v^{-2}}{\Delta_t} p_V - \frac{B_{t,23}^T + \sigma_v^{-2}}{\Delta_t} p_N \right) B_{t,13}^T = 0,
\]

\[
\frac{1}{T} \sum_{t=0}^{T} \left( 1 - (R^t)^{t-T} \right) \left[ \frac{1 + (\rho_V - R^t) p_V - \frac{B_{t,33}^T + \sigma_N^{-2}}{\Delta_t} p_N - \frac{B_{t,23}^T + \sigma_v^{-2}}{\Delta_t} p_V}{\Delta_t} \rho_V B_{t,22}^T \right] = 0,
\]

\[
p_N \frac{1}{T} \sum_{t=0}^{T} \left( 1 - (R^t)^{t-T} \right) \left[ \frac{1 + (\rho_V - R^t) p_V - \frac{B_{t,33}^T + \sigma_N^{-2}}{\Delta_t} p_N - \frac{B_{t,23}^T + \sigma_v^{-2}}{\Delta_t} p_V}{\Delta_t} \rho_V B_{t,23}^T \right] = \frac{R^t - 1}{R^t \gamma}.
\]
Conjecture a stationary solution in which:

\[1 + (\rho_V - R^f)p_V - \left(\frac{B_{t,33}^T + \sigma_N^{-2}p_V}{\Delta_t} - \frac{B_{t,32}^T}{\Delta_t}p_N\right) \rho_V B_{t,22} + \left(- \frac{B_{t,22}^T + \sigma_N^{-2}p_N}{\Delta_t} + \frac{B_{t,23}^T}{\Delta_t}p_V\right) \rho_V B_{t,23}^T = 0,\]

for each \(t\), then it then follows that the lower right minor of \(B_t^T\) satisfies the recursion:

\[R^f \begin{bmatrix} B_{t-1,22}^T & B_{t-1,23}^T \\ B_{t-1,32}^T & B_{t-1,33}^T \end{bmatrix} = \begin{bmatrix} B_{t,22}^T \left(1 - \frac{B_{t,33}^T + \sigma_N^{-2}B_{t,22}^T}{\Delta_t}\right) \rho_V^2 & 0 \\ - \left(\frac{B_{t,22}^T + \sigma_N^{-2}B_{t,23}^T}{\Delta_t} - \frac{B_{t,32}^T}{\Delta_t}B_{t,22}^T\right) \rho_V^2 B_{t,23}^T & 0 \end{bmatrix},\]

from which follows that:

\[B_{t,23}^T = B_{t,32}^T = 0.\]

Consequently, the above further reduces to:

\[R^f \begin{bmatrix} B_{t-1,22}^T & 0 \\ 0 & B_{t-1,33}^T \end{bmatrix} = \begin{bmatrix} B_{t,22}^T \left(1 - \frac{B_{t,33}^T + \sigma_N^{-2}B_{t,22}^T}{\Delta_t}\right) \rho_V^2 & 0 \\ 0 & \left(R^f \rho_N\right)^2 \frac{\sigma_D^2 + \frac{B_{t,33}^T + \sigma_N^{-2}B_{t,22}^T}{\Delta_t}^2}{\left(R^f \rho_N\right)^2 + \frac{B_{t,22}^T + \sigma_N^{-2}B_{t,23}^T}{\Delta_t}^2 - \frac{B_{t,22}^T + \sigma_N^{-2}B_{t,23}^T}{\Delta_t} p_N} \right)\]

and we see that:

\[B_{t-1,22}^T = B_{t,22}^T \left(1 - \frac{B_{t,33}^T + \sigma_N^{-2}B_{t,22}^T}{\Delta_t}\right) \rho_V^2 = \frac{\sigma_V^{-2}B_{t,22}^T}{\sigma_V^{-2}B_{t,22}^T + \sigma_V^{-2}p_V^2} B_{t,23}^T = 0,\]

since \(B_{t,22}^0 = 0\), from which follows that:

\[p_V = \frac{1}{R^f - \rho_V},\]

and consequently:

\[B_{t-1,33}^T = \frac{R^f \rho_N^2}{\sigma_D^2 + \left(\frac{\sigma_V}{R^f - \rho_V}\right)^2 + \frac{\rho_N^2}{B_{t,33}^T + \sigma_N^{-2}}} .\]

Since the recursion for \(B_t^T\) does not depend on time-varying objects, we drop the \(t\) subscript and index \(B_t^T\) only by the investor’s remaining horizon, \(B_{33}^T\). Since all coefficients on the RHS are positive, that \(B_{t-1,33}^T > B_{t,33}^T \forall t\). It then follows that:

\[(\Xi^T B_t^T \Xi + \Omega^{-1})^{-1} = \begin{bmatrix} \sigma_D^2 & 0 & 0 \\ 0 & \sigma_V^2 & 0 \\ 0 & 0 & \frac{1}{B_{33}^T + \sigma_N^{-2}} \end{bmatrix},\]

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which is always positive definite since $B_{33}^T \geq 0$.

In addition, the asset demand of investor of generation $t$’s asset demand is given by:

$$X_t^T = -\frac{R^f 1 - (R^f)^{t-T}}{\gamma} \frac{R^f p_N N_t}{\sigma_D^2 + \left(\frac{\sigma_v}{R^f - \rho_v}\right)^2 + \frac{\sigma_v^2}{B_{33}^T + \sigma_N^2}}. $$

Substituting this demand into market-clearing along with the definition of $B_{33}^T$, we arrive at:

$$\frac{1}{T} \sum_{t=0}^{T} \left(1 - (1/R^f)^{T-t}\right) B_{33}^{T-1} = \frac{R^f - 1}{R^f} \frac{\gamma p_N}{\gamma}. $$

Consider now a more conservative economy in which everyone is of the investor cohort with the maximum horizon of $\tau$. Then:

$$B_{33}^\tau = \frac{R^f - 1}{1 - (1/R^f)^T} \frac{\gamma p_N}{R^f}. $$

This implies the market-clearing condition for $N_t$ can be expressed as:

$$\frac{p_N^2 \sigma_N^2}{1 + B_{33}^\tau \sigma_N^2} - \frac{1 - (1/R^f)^T}{1 - R^f} \frac{(R^f)^2}{\gamma} p_N + \sigma_D^2 + \left(\frac{\sigma_v}{R^f - \rho_v}\right)^2 = 0. $$

Consider case that $\tau = 2$, then $p_N$ solves the quartic polynomial:

$$0 = p_N^4 \sigma_N^4 - \frac{(1 + R^f)^2}{\gamma} p_N^2 \sigma_N^2 + (2 + R^f) \left(\sigma_D^2 + \left(\frac{\sigma_v}{R^f - \rho_v}\right)^2\right) p_N^2 \sigma_N^2 $$

$$- \left(\sigma_D^2 + \left(\frac{\sigma_v}{R^f - \rho_v}\right)^2\right) \frac{1 + R^f}{\gamma} p_N + \left(\sigma_D^2 + \left(\frac{\sigma_v}{R^f - \rho_v}\right)^2\right)^2. $$

By Descartes’ Rule of Signs, this polynomial has zero positive real roots and either zero, two, or four positive real roots. This quartic polynomial has no real solutions for $\gamma$ sufficiently large, as the limiting polynomial as $\gamma \to \infty$ is:

$$0 = p_N^4 \sigma_N^4 + (2 + R^f) \left(\sigma_D^2 + \left(\frac{\sigma_v}{R^f - \rho_v}\right)^2\right) p_N^2 \sigma_N^2 + \left(\sigma_D^2 + \left(\frac{\sigma_v}{R^f - \rho_v}\right)^2\right)^2, $$

which has no real roots. Consequently, when investors have a two-period horizon, the market can still break down if risk aversion is sufficiently large. The issue is even more severe in the true economy, since half of the investors have a shorter one-period horizon, and less risk-bearing tolerance than the two-period investors.
Consider instead the limit that $\tau \to \infty$, and investors have an infinite horizon. Then, $B_{33}^{\infty}$ can be solved as the fixed-point of the recursion for $B_{33}^{T}$, and $p_N$ solves the fixed-point problem from market-clearing:

$$\frac{R_f - 1}{(R_f)^2} \gamma \left( \sigma_D^2 + \left( \frac{\sigma_V}{R_f - p_V} \right)^2 + \frac{p_N^2}{-\frac{R_f - 1}{R_f} \gamma p_N + \sigma_N^2} \right) = p_N.$$ 

It follows that $p_N$ is then given explicitly by:

$$p_N = \frac{\gamma \left( \sigma_D^2 + \frac{1}{R_f} \left( \frac{\sigma_V}{1 - p_V} \right)^2 \right) - \sigma_N^2 \frac{R_f}{\gamma}}{2 (R_f - 1)} + \sqrt{\left( \frac{\sigma_N^2 \frac{R_f}{\gamma} - \gamma \left( \sigma_D^2 + \frac{1}{R_f} \left( \frac{\sigma_V}{1 - p_V} \right)^2 \right)}{2 (R_f - 1)} \right)^2 + \frac{\sigma_N^2 \left( \sigma_D^2 + \frac{1}{R_f} \left( \frac{\sigma_V}{1 - p_V} \right)^2 \right)}{R_f - 1}},$$

which always exists.

Consequently, it follows that for short horizon investors, markets can break down because investors are too risk averse, while with infinite horizon investors, a market equilibrium always exists.

**Proof of Proposition 3**

We derive the perfect information equilibrium with trading by the government. We first conjecture that, when $V_{t+1}$ and $N_t$ are observable to the government and investors, the stock price takes the linear form:

$$P_t = p_V V_{t+1} + p_N N_t + p_g G_t.$$ 

Given that dividends are $D_t = V_t + \sigma_D \varepsilon_t^D$, the stock price must react to a deterministic unit shift in $V_{t+1}$ by the present value of dividends deriving from that shock, $\frac{1}{R_f - p_V}$, it follows that $p_V = \frac{1}{R_f - p_V}$. The innovations to $V_{t+1}$ and $N_t$ are the only source of risk and, from the perspective of all economic agents, the conditional expectation and variance of $R_{t+1}$ are:

$$E \left[ R_{t+1} \mid \mathcal{F}_t \right] = -p_N R_f N_t - R_f p_g G_t,$$

$$Var \left[ R_{t+1} \mid \mathcal{F}_t \right] = \sigma_D^2 + \left( \frac{\sigma_V}{R_f - p_V} \right)^2 + p_N^2 \sigma_N^2 + p_g^2 \sigma_G^2.$$ 

Since all investors are identical when $V_t$ and $N_t$ are observable, it follows that in the CARA-Normal environment all investors have an identical mean-variance demand for the risky
asset:
\[
X_t^s = \frac{1}{\gamma} \frac{E[R_{t+1} | \mathcal{F}_t]}{Var [R_{t+1} | \mathcal{F}_t]} = \frac{-1}{\gamma} \frac{p_N R^f N_t + R^f p_g G_t}{\sigma^2_D + \left( \frac{\sigma_V}{R^f - \rho_V} \right)^2 + p_N^2 \sigma^2_N + p_g^2 \sigma^2_G}.
\]

In the government’s intervention rule:
\[
X_t^G = -\vartheta_N N_t + \vartheta_N \sigma_N G_t,
\]

Finally, by imposing market-clearing, we arrive at:
\[
N = \frac{1}{\gamma} \frac{p_N R^f N}{\sigma^2_D + \left( \frac{\sigma_V}{R^f - \rho_V} \right)^2 + p_N^2 \sigma^2_N + p_g^2 \sigma^2_G} + \vartheta_N N,
\]
\[
\vartheta_N \sigma^2_N G_t = \frac{1}{\gamma} \frac{R^f p_g}{\sigma^2_D + \left( \frac{\sigma_V}{R^f - \rho_V} \right)^2 + p_N^2 \sigma^2_N + p_g^2 \sigma^2_G} G_t
\]

which, by matching coefficients, reveals that:
\[
\frac{1}{\gamma} \frac{p_N R^f}{\sigma^2_D + \left( \frac{\sigma_V}{R^f - \rho_V} \right)^2 + p_N^2 \sigma^2_N + p_g^2 \sigma^2_G} + \vartheta_N = 1,
\]
\[
\vartheta_N \sigma_N = p_g.
\]

This confirms the conjectured equilibrium.

Rearranging this equation for \( p_N \), and substituting for \( p_g \), we arrive at the quadratic equation for \( p_N \):
\[
\left( 1 + \left( \frac{\vartheta_N}{1 - \vartheta_N} \right)^2 \sigma^2_G \right) p_N^2 - \frac{R^f}{\gamma \sigma^2_N (1 - \vartheta_N) p_N} + \frac{\sigma^2_D}{\vartheta_N^2} + \left( \frac{\sigma_V}{R^f - \rho_V} \right)^2 \frac{1}{\sigma^2_N} = 0,
\]

from which follows that \( p_N \) has two roots:
\[
p_N (\vartheta_N) = \frac{1}{2} \frac{R^f}{\gamma \sigma^2_N (1 - \vartheta_N)} \pm \sqrt{\frac{R^f}{\gamma \sigma^2_N (1 - \vartheta_N)} - 4 \left( 1 + \left( \frac{\vartheta_N}{1 - \vartheta_N} \right)^2 \sigma^2_G \right) \left( \frac{\sigma^2_D}{\vartheta_N^2} + \left( \frac{\sigma_V}{R^f - \rho_V} \right)^2 \frac{1}{\sigma^2_N} \right)}.
\]

Recognizing that two positive solutions for \( p_N \) exist if the expression under the square root is nonnegative, it follows that the market breaks down occurs whenever:
\[
R^f < 2 (1 - \vartheta_N) \gamma \sqrt{\left( 1 + \left( \frac{\vartheta_N}{1 - \vartheta_N} \right)^2 \sigma^2_G \right) \left( \sigma^2_D \sigma^2_N + \left( \frac{\sigma_V}{R^f - \rho_V} \right)^2 \sigma^2_N \right)}.
\]
Consequently, market breakdown occurs when $\sigma_N$ is sufficiently large.

Given that:

\[ Var (\Delta P_t | \mathcal{F}_{t-1}) = Var \left( P_{t+1} - \frac{1}{Rf} - \rho \sigma_P V_{t+1} | \mathcal{F}_{t-1} \right) \]

\[ = \left( \frac{\sigma_P}{Rf - \rho} \right)^2 + \rho^2 \sigma^2 + p^2 \sigma^2_G \]

\[ = \left( 1 + \left( \frac{\varrho_n}{1 - \varrho_n} \right)^2 \right) \rho^2 \sigma^2_N, \]

substituting for $p_g$, it follows that regardless of whether the government is concerned with price volatility or price informativeness, reducing the price variance from noise trading, $p^2 \sigma^2_N$, would accomplish both objectives since:

\[ p^2 \sigma^2_N = \frac{-Rf}{\gamma(1-\varrho_n)} + \rho^2 \sigma^2_D + \left( \frac{\sigma_P}{Rf - \rho} \right)^2 \]

\[ 1 + \left( \frac{\varrho_n}{1 - \varrho_n} \right)^2 \sigma^2_G, \]

is increasing in $\sigma^2_N$ through $p_N$.

To establish that the linear equilibrium is the unique, symmetric equilibrium, we express each investor’s optimization problem as:

\[ U_t = \max_{X_t} E \left[ e^{-\gamma (R\tilde{W} + X_t (V_{t+1} + \sigma_D \varepsilon_{t+1} + P_{t+1} - Rf))} | \mathcal{F}_t \right] \]

For an arbitrary price function $P_t$, the FOC for the investor’s holding of the risky asset $X_t$ is:

\[ E \left[ (V_{t+1} + \sigma_D \varepsilon_{t+1} + P_{t+1} - Rf P_t) e^{-\gamma X_t (V_{t+1} + \sigma_D \varepsilon_{t+1} + P_{t+1} - Rf)} | \mathcal{F}_t \right] = 0. \]

Substituting this with the market-clearing condition:

\[ X_t = - (1 - \varrho_n) N_t - \varrho_n \sigma_N G_t, \]

we arrive at:

\[ E \left[ (V_{t+1} + \sigma_D \varepsilon_{t+1} + P_{t+1} - Rf P_t) e^{\gamma ((1-\varrho_n) N_t + \varrho_n \sigma_N G_t)(V_{t+1} + \sigma_D \varepsilon_{t+1} + P_{t+1} - Rf)} | \mathcal{F}_t \right] = 0. \]

Since $P_{t+1}$ cannot be a function of $\varepsilon_{t+1}$, as $P_{t+1}$ is forward-looking for the new generation of investors at time $t + 1$, the above can be rewritten as:

\[ P_t = \frac{1}{Rf} V_{t+1} + \frac{\gamma}{Rf} \sigma^2_D ((1 - \varrho_n) N_t + \varrho_n \sigma_N G_t) + \frac{1}{Rf} E \left[ P_{t+1} e^{-\gamma ((1-\varrho_n) N_t + \varrho_n \sigma_N G_t) P_{t+1}} | \mathcal{F}_t \right], \] (IA.2)
where we have used the properties of log-normal random variables to complete the square in the pdf and solve explicitly for the $\varepsilon_{t+1}^D$ term. This defines a functional equation, whose fixed point is the price functional $P_t$. To see that the linear equilibrium we derived above solves this functional equation, we rewrite equation (IA.2) as:

$$P_t = \frac{1}{R^f} V_{t+1} + \frac{\gamma}{R^f} \sigma_D^2 ((1 - \vartheta_N) N_t + \vartheta_N \sigma_N G_t) + \left(\frac{1}{R^f}\right) \partial_u \log E \left[ e^{uP_{t+1}} \mid \mathcal{F}_t \right] \bigg|_{u=-\gamma((1-\vartheta_N)N_t+\vartheta_N\sigma_N G_t)} ,$$

and conjecture that $P_t = \frac{1}{R^f-\rho_V} V_{t+1} + p_N N_t + p_G G_t$, from which follows that $p_N$ satisfies the recursion:

$$p_{N,t} = \frac{\gamma (1 - \vartheta_N)}{R^f} \left( \sigma_D^2 + \left( \frac{\sigma_V}{R^f - \rho_V} \right)^2 \left( 1 + \left( \frac{\vartheta_N}{1 - \vartheta_N} \right)^2 \sigma_G^2 \right) \sigma_N^2 p_{N,t+1}^2 \right).$$

Suppose there is some final date $T >> 0$. On this final date, $P_T = 0$ since there is no salvage value to the asset. Then, as time goes backward, this recursion converges after a sufficiently long period of time to:

$$p_{N,t} \to_{t \to 0} \frac{1}{2} \frac{1}{\gamma \sigma_N^2 (1-\vartheta_N)} - \sqrt{\left( \frac{\gamma \sigma_N^2 (1-\vartheta_N)}{R^f} \right)^2 - 4 \left( 1 + \left( \frac{\vartheta_N}{1 - \vartheta_N} \right)^2 \sigma_G^2 \right) \left( \sigma_D^2 + \left( \frac{\sigma_V}{R^f - \rho_V} \right)^2 \frac{1}{\sigma_N^2} \right) \left( 1 + \left( \frac{\vartheta_N}{1 - \vartheta_N} \right)^2 \sigma_G^2 \right)},$$

which is the more stable of the two positive roots from the infinite horizon problem if:

$$R^f < 2 (1 - \vartheta_N) \gamma \sqrt{\left( 1 + \left( \frac{\vartheta_N}{1 - \vartheta_N} \right)^2 \sigma_G^2 \right) \left( \sigma_D^2 \sigma_N^2 + \left( \frac{\sigma_V}{R^f - \rho_V} \right)^2 \sigma_N^2 \right)},$$

and:

$$p_{N,t} \to_{t \to 0} \infty,$$

otherwise. Consequently, we can interpret market breakdown as an unstable backward recursion in which illiquidity is growing each period as volatility diverges. Interestingly, we obtain the more positive root for the fixed point for $p_N$ from (IA.1) from the forward recursion:

$$p_{N,t+1} = \frac{\gamma (1 - \vartheta_N)}{R^f} \left( \sigma_D^2 + \left( \frac{\sigma_V}{R^f - \rho_V} \right)^2 \left( 1 + \left( \frac{\vartheta_N}{1 - \vartheta_N} \right)^2 \sigma_G^2 \right) \sigma_N^2 p_{N,t+1}^2 \right).$$

Consequently, the more positive root is forward stable, but backward unstable.

Finally, notice from the recursion (IA.2) that, if the price at date $t+1$, $P_{t+1}$, is linear in $\{V_{t+1}, N_{t+1}, G_{t+1}\}$, and therefore normally distributed, then $\log E \left[ e^{uP_{t+1}} \mid \mathcal{F}_t \right] =
\[ uE[P_{t+1} | \mathcal{F}_t] + \frac{1}{2}u^2 \text{Var}[P_{t+1} | \mathcal{F}_t], \] or the moment-generating function for the normally distributed price. It then follows that the only solution is

\[ P_t = \frac{1}{R_f} V_{t+1} + \gamma \frac{\sigma_D^2}{R_f} ((1 - \varpi_N) N_t + \varpi_N \sigma_N G_t) + E[P_{t+1} | \mathcal{F}_t] \]

\[ + \gamma \frac{\sigma_D^2}{R_f} ((1 - \varpi_N) N_t + \varpi_N \sigma_N G_t) \text{Var}[P_{t+1} | \mathcal{F}_t], \]

and it follows that \( P_t \) is linear. Consequently, the linear equilibrium with the less positive \( \varpi_N \) root of (IA.1) is the unique, backward stable equilibrium as the limit of the finite horizon problem.

**Proof of Proposition A1**

Based on Proposition A3 and Proposition A5, in the special case of no government intervention, the steady-state conditional means of the Kalman Filter, \( (\hat{V}_{t+1}^M, \hat{N}_{t+1}^M) \), have a law of motion that satisfies:

\[
\begin{bmatrix}
\hat{V}_{t+1}^M \\
\hat{N}_{t+1}^M
\end{bmatrix} =
\begin{bmatrix}
\rho_V & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{V}_{t}^M \\
\hat{N}_{t-1}^M
\end{bmatrix}
+ k_t^M
\begin{bmatrix}
D_t - \hat{V}_{t}^M \\
\eta_t^M - \rho_V \hat{V}_{t}^M
\end{bmatrix},
\]

where:

\[
k_t^M = \begin{bmatrix}
\rho_V \Sigma^M, VV & p_V (\rho_V^2 \Sigma^M, VV + \sigma_V^2) \\
0 & p_{NN} \sigma_N^2
\end{bmatrix} \left( \Omega^M \right)^{-1}
\]

is the Kalman Gain, and the conditional variance \( \Sigma^M \) satisfies the Ricatti Equation:

\[
\Sigma^M = \begin{bmatrix}
\rho_V & 0 \\
0 & 0
\end{bmatrix} \Sigma^M
+ \begin{bmatrix}
\rho_V & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\rho_V & 0 \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
\sigma_V^2 & 0 \\
0 & \sigma_N^2
\end{bmatrix}
- k_t^M
\begin{bmatrix}
\rho_V \Sigma^M, VV \\
0
\end{bmatrix} p_V \left( \rho_V \Sigma^M, VV + \sigma_V^2 \right),
\]

and that:

\[
\Omega^M = \text{Var}
\begin{bmatrix}
D_t - \hat{V}_{t}^M \\
\eta_t^M - \rho_V \hat{V}_{t}^M
\end{bmatrix}
\mid \mathcal{F}_{t-1}^M
\]

\[
= \begin{bmatrix}
\Sigma^M, VV + \sigma_D^2 \\
p_{NN} \sigma_N^2
\end{bmatrix}
\begin{bmatrix}
p_V \rho_V \Sigma^M, VV \\
p_{NN} \sigma_N^2
\end{bmatrix} p_V \left( \rho_V \Sigma^M, VV + \sigma_V^2 \right) + p_{NN}^2 \sigma_N^2
\].

From Proposition A4, we further recognize that:

\[
\Sigma^M, VN = - \frac{p_V}{p_N} \Sigma^M, VV,
\]

\[
\Sigma^M, NN = \left( \frac{p_V}{p_N} \right)^2 \Sigma^M, VV.
\]
Consequently, recognizing that all four implied coefficient equations for Σ^M are degenerate, the equation that identifies Σ^{M,VV} reduces to:

\[
\Sigma^{M,VV} = \frac{\sigma_v^2}{\frac{\rho_N \sigma_N}{\rho_V \sigma_V}} + \frac{\rho^2 \left( \frac{\sigma_D}{\sigma_V} \right)^2}{\sigma_v^2} + \frac{\rho^2 \left( \frac{\sigma_D}{\sigma_V} \right)^2}{\sigma_v^2} + \frac{\rho^2 \left( \frac{\sigma_D}{\sigma_V} \right)^2}{\sigma_v^2}.
\]

From Proposition A4, investors update their beliefs from the market beliefs by Bayes’ Law in accordance with a linear updating rule. The posterior of investor i is \( N(\hat{V}_{t+1}^i, \Sigma^i_s) \), where \( \hat{V}_{t+1}^i = E[V_{t+1} \mid F_t] \) and \( \Sigma^i_s = E \left[ (V_{t+1} - \hat{V}_{t+1}^i)^2 \mid F_t \right] \) are given by:

\[
\hat{V}_{t+1}^i = \hat{V}_{t+1}^M + \frac{\Sigma^{M,VV}}{\Sigma^{M,VV} + \tau^{-1}_s} (s^i_t - \hat{V}_{t+1}^M),
\]

and:

\[
\Sigma^i_s = (\Sigma^{M,VV})^{-1} + \tau_s.
\]

This characterizes the beliefs of investors given the market beliefs.

Since the government does not trade in this benchmark, investors have no incentive to learn about the government’s behavior, and therefore the information acquisition decision is trivial. Given that investors each acquire a private signal \( s^i_t \), standard results for CARA utility with normally distributed prices and payoffs establish that the optimal trading policy of investor i, \( X^i_t \), is given by:

\[
X^i_t = \frac{E [D_{t+1} + P_{t+1} - R^f P_t \mid F_t]}{\gamma \text{Var} [D_{t+1} + P_{t+1} \mid F_t]} \left( 1 + p_V \left( \rho_V - R^f \right) \right) \left( \hat{V}_{t+1}^i - \hat{V}_{t+1}^M \right) - p_N R^f \hat{N}_{t}^i.
\]

where:

\[
\varphi = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left( \frac{\rho_V - p_V}{0} \right) \kappa^M \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right),
\]

and:

\[
\Omega(i) = \Omega^M - \frac{1}{\rho_V} \left( \frac{\Sigma^{M,VV}}{\Sigma^{M,VV} + \tau^{-1}_s} \right) \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).
\]

is the conditional variance of \( D_{t+1} \) and \( P_{t+1} \) with respect to \( F_t^i \). We can rewrite the above as:

\[
X^i_t = \frac{\left( 1 + p_V \left( \rho_V - R^f \right) \right) \left( \hat{V}_{t+1}^i - \hat{V}_{t+1}^M \right) - p_N R^f \hat{N}_{t}^i}{\gamma \varphi' \Omega(i) \varphi}.
\]
Substituting for \( \hat{V}_{i+1} \), and recognizing from above that:

\[
\hat{N}_t^M = N_t + \frac{p_V}{p_N} \left( V_{t+1} - \hat{V}_{t+1}^M \right),
\]

and therefore that:

\[
\hat{N}_i = \hat{N}_t^M - \frac{p_V}{p_N} \left( \hat{V}_{i+1} - \hat{V}_{t+1}^M \right) = N_t + \frac{p_V}{p_N} \left( V_{i+1} - \hat{V}_{t+1}^M \right) - \frac{p_V}{p_N} \left( \hat{V}_{i+1} - \hat{V}_{t+1}^M \right),
\]

we arrive at:

\[
X_i^t = \left( \varphi' \left[ \frac{1}{p_V \rho_V} \right] \frac{\sum_{i=M}^{V_i} \left( s_{i} - \hat{V}_{i+1}^M \right) - R^f p_N N_t - R^f p_V \left( V_{i+1} - \hat{V}_{t+1}^M \right) \gamma \varphi \Omega (i) \varphi \right] \right).
\]

Aggregating over the demand of investors and imposing market-clearing, we arrive at the two equations for \( p_V \) and \( p_N \):

\[
\varphi' \left[ \frac{1}{p_V \rho_V} \right] \sum_{V,V}^{M} \frac{R^f p_N \gamma \varphi \Omega (i) \varphi}{\sum_{V,V}^{M} + \tau^{-1}} = 0,
\]

\[
\frac{R^f p_N \gamma \varphi \Omega (i) \varphi}{\sum_{V,V}^{M} + \tau^{-1}} = 1.
\]

This completes our characterization of the linear equilibrium.

We now recognize from the market-clearing condition for \( p_V \) that we can express \( p_V \) as

\[
p_V = \left( \frac{1}{R^f - \rho_V} + \sum_{V,V}^{M} \frac{\rho_V^2 (1 - \rho_V)}{\sigma_V} \left( \frac{\sum_{V,V}^{M} \rho_V}{\sigma_V} + \left( \frac{\sigma_D}{\sigma_V} \right)^2 - \left( \frac{\rho_N \sigma_N}{p_V \sigma_V} \right)^2 \left( \frac{\sigma_D}{\sigma_V} \right)^2 \right) \right) \sum_{V,V}^{M} \frac{R^f p_N \gamma \varphi \Omega (i) \varphi}{\sum_{V,V}^{M} + \tau^{-1}}.
\]

From our implicit equation for \( \Sigma_{V,V}^{M} \) above, we can verify that the second term in parentheses is negative (assuming \( p_V > p_V \)), from which follows that

\[
p_V \leq \frac{1}{R^f - \rho_V \sum_{V,V}^{M} + \tau^{-1}} \leq \frac{1}{R^f - \rho_V} = p_V,
\]

confirming the assumption.

Finally, recognizing that return volatility from the market perspective satisfies

\[
\varphi' \Omega^M \varphi = \left( 1 + 2 p_V \rho_V + p_V^2 \rho_V^2 + 2 (p_V - p_V) p_V \rho_V - (p_V - p_V)^2 (1 - \rho_V^2) \right) \sum_{V,V}^{M} \sigma_D + p_N^2 \sigma_N^2 + p_V^2 \sigma_V^2
\]

which makes use of the relation by the Law of Total Variance

\[
k^M \Omega^M k^M = \left[ \begin{array}{cc}
\rho_V^2 \sum_{V,V}^{M} & \sigma_V^2 \\
0 & \sigma_N^2
\end{array} \right] - \Sigma^M.
\]
we can rewrite the market-clearing condition for $p_N$, substituting with that of $p_V$, as

\[
\frac{Rf p_N}{\gamma} = \left(1 + 2(p_V + p_V p_V - p_V^2) \rho_V + p_V^2 \rho_V^2 - (p_V - p_V)^2 - \frac{\Sigma M, VV + \tau^{-1}_s}{\Sigma M, VV} (Rf)^2 \rho_V^2 \right) \sum M, VV + \sigma^2_D + p_N^2 \sigma^2_V + p_V^2 \rho_V^2.
\]

Notice in the special case that $\rho_V = 0$ that

\[
p_V = \frac{\sum M, VV}{\sum M, VV + \tau^{-1}_s} \frac{1}{Rf},
\]

and the above condition for $p_N$ reduces to

\[
\frac{Rf p_N}{\gamma} = \left(1 - \left(\frac{1}{Rf}\right)^2 \frac{\tau^{-1}_s}{\Sigma M, VV + \tau^{-1}_s}\right) \frac{\sum M, VV \tau^{-1}_s}{\Sigma M, VV + \tau^{-1}_s} + \sigma^2_D + \left(\frac{1}{Rf}\right)^2 \sigma^2_V + p_N^2 \sigma^2_V,
\]

since $p_V = \frac{1}{Rf}$. Comparing this condition for $p_N$ to the perfect information case

\[
\frac{1}{\gamma} p_N Rf = \sigma^2_D + \left(\frac{1}{Rf}\right)^2 \sigma^2_V + p_N^2 \sigma^2_V,
\]

we recognize since $\frac{\sum M, VV \tau^{-1}_s}{\Sigma M, VV + \tau^{-1}_s} > 0$ that the additional term from uncertainty exacerbates the market breakdown problem. To see this, we fix $\sum M, VV$ and express the solution to $p_N$ as

\[
p_N = \frac{1}{2\sigma^2_N} A - \sqrt{\left(\frac{1}{2\sigma^2_N} A\right)^2 - \frac{1}{\sigma^2_N} B},
\]

where:

\[
A = \frac{Rf}{\gamma},
\]

\[
B = \sigma^2_D + \left(\frac{1}{Rf}\right)^2 \sigma^2_V + \left(1 - \left(\frac{1}{Rf}\right)^2 \frac{\tau^{-1}_s}{\Sigma M, VV + \tau^{-1}_s}\right) \frac{\sum M, VV \tau^{-1}_s}{\Sigma M, VV + \tau^{-1}_s}.
\]

Since $\left(1 - \frac{1}{Rf^2} \frac{\tau^{-1}_s}{\Sigma M, VV + \tau^{-1}_s}\right) \frac{\sum M, VV \tau^{-1}_s}{\Sigma M, VV + \tau^{-1}_s} > 0$, regardless of the equilibrium value of $p_N$, it follows nonexistence, which occurs when

\[
\left(\frac{1}{2\sigma^2_N} A\right)^2 - \frac{1}{\sigma^2_N} B < 0,
\]

must now occur at a positive value of $p_N$, and that $p_N$ is higher in the presence of informational frictions when a solution exists (by shrinking the $\sqrt{\left(\frac{1}{2\sigma^2_N} A\right)^2 - \frac{1}{\sigma^2_N} B}$ term in the expression for $p_N$). From the condition for existence in Proposition 1, it follows that market-breakdown must occur at a lower value of $\sigma_N^*, \sigma_N^{**} \geq \sigma_N^*$. 

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Consider the other extreme of $\rho_V = 1$, then the coefficient on $\Sigma^{M, VV}$ in the expression for $p_N$ reduces to

$$1 + 2p_V + (2p_V - p_V)p_V - \frac{\Sigma^{M, VV} + \tau_s^{-1}}{\Sigma^{M, VV}} (R^f)^2 p_V^2.$$  

Since $p_V \leq \frac{\Sigma^{M, VV}}{\Sigma^{M, VV} + \tau_s^{-1}} p_V$ and with $p_V = \frac{1}{R^f - 1}$, it follows that

$$1 + 2p_V + (2p_V - p_V)p_V - \frac{\Sigma^{M, VV} + \tau_s^{-1}}{\Sigma^{M, VV}} (R^f)^2 p_V^2 > 0,$$

and consequently similar arguments establish that market breakdown occurs sooner, and $p_N$ is more positive with informational frictions. The intermediate cases ($\rho_V \in (0, 1)$) follow by similar arguments.

**Proof of Proposition A2**

Consider now conditional price volatility

$$Var \left[ \Delta P_{t+1} \mid \mathcal{F}_t^M \right] = \left( \varphi - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)' \Omega^M \left( \varphi - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = (p_V^2 \rho_N^2 + 2(p_V - p_V)p_V \rho_V - (p_V - p_V)^2 (1 - \rho_V^2)) \Sigma^{M, VV}$$

$$+ p_V^2 \sigma_V^2 + p_N^2 \sigma_N^2.$$  

In the special case that $\rho_V = 0$ that

$$Var \left[ \Delta P_{t+1} \mid \mathcal{F}_t^M \right] = p_V^2 \sigma_V^2 + p_N^2 \sigma_N^2 - (p_V - p_V)^2 \Sigma^{M, VV}$$

$$= p_V^2 \sigma_V^2 + p_N^2 \sigma_N^2 - \left( \frac{1}{R^f} \right)^2 \left( \frac{\tau_s^{-1}}{\Sigma^{M, VV} + \tau_s^{-1}} \right)^2 \Sigma^{M, VV}.$$  

From the market-clearing condition for $p_N$, one has that

$$p_N^2 \sigma_N^2 - \left( \frac{1}{R^f} \right)^2 \left( \frac{\tau_s^{-1}}{\Sigma^{M, VV} + \tau_s^{-1}} \right)^2 \Sigma^{M, VV} = \frac{R^f p_N}{\gamma} - \frac{\Sigma^{M, VV} \tau_s^{-1}}{\Sigma^{M, VV} + \tau_s^{-1}} - \sigma_D^2 - \left( \frac{1}{R^f} \right)^2 \sigma_V^2,$$

the above can be rewritten as

$$Var \left[ \Delta P_{t+1} \mid \mathcal{F}_t^M \right] = p_V \sigma_V^2 + \frac{R^f p_N}{\gamma} - \frac{\Sigma^{M, VV} \tau_s^{-1}}{\Sigma^{M, VV} + \tau_s^{-1}} - \sigma_D^2 - \left( \frac{1}{R^f} \right)^2 \sigma_V^2.$$  

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Subtracting price volatility under perfect information, \( p_V \sigma_V^2 + \tilde{p}_N \sigma_N^2 \), where \( \tilde{p}_N \) is the coefficient on \( N_t \) under perfect information, and recognizing that
\[
\tilde{p}_N \sigma_N^2 = \frac{1}{\gamma} \tilde{p}_N R' - \sigma_D^2 - \left( \frac{1}{R'} \right)^2 \sigma_V^2,
\]
from its market-clearing condition, we arrive at
\[
Var \left[ \Delta P_{t+1} \mid F_t^M \right] - p_V \sigma_V^2 - \tilde{p}_N \sigma_N^2
\]
\[
= \frac{R'}{\gamma} (p_N - \tilde{p}_N) - \frac{\sum_{M, VV} \tau^{-1}_s}{\sum_{M, VV} + \tau^{-1}_s}
\]
\[
= - \frac{\sum_{M, VV} \tau^{-1}_s}{\sum_{M, VV} + \tau^{-1}_s} + p_V^2 \sigma_V^2 + \tilde{p}_N^2 \sigma_N^2 + \frac{R'}{\gamma \sigma_N} \sqrt{\frac{1}{4} \left( \frac{R'}{\tau^{-1}_s} \right)^2 - \sigma_D^2 - \left( \frac{1}{R'} \right)^2 \sigma_V^2}
\]
\[-\frac{R'}{\gamma \sigma_N} \frac{1}{4} \left( \frac{R'}{\tau^{-1}_s} \right)^2 - \sigma_D^2 - \left( \frac{1}{R'} \right)^2 \sigma_V^2 - \left( 1 - \left( \frac{1}{R'} \right)^2 \frac{\tau^{-1}_s}{\sum_{M, VV} + \tau^{-1}_s} \right) \frac{\sum_{M, VV} \tau^{-1}_s}{\sum_{M, VV} + \tau^{-1}_s}
\]
It then follows from a first-order Taylor expansion of \(-\sqrt{x - a}\) around \( a = 0\), (which omits a positive residual when \( a \) is less than 1)
\[
-\sigma_V \sqrt{\frac{1}{4} \left( \frac{R'}{\tau^{-1}_s} \right)^2 - \sigma_D^2 - \left( \frac{1}{R'} \right)^2 \sigma_V^2} - \left( 1 - \left( \frac{1}{R'} \right)^2 \frac{\tau^{-1}_s}{\sum_{M, VV} + \tau^{-1}_s} \right) \frac{\sum_{M, VV} \tau^{-1}_s}{\sum_{M, VV} + \tau^{-1}_s}
\]
\[-\frac{R'}{\gamma \sigma_N} \frac{1}{4} \left( \frac{R'}{\tau^{-1}_s} \right)^2 - \sigma_D^2 - \left( \frac{1}{R'} \right)^2 \sigma_V^2 + \left( 1 - \left( \frac{1}{R'} \right)^2 \frac{\tau^{-1}_s}{\sum_{M, VV} + \tau^{-1}_s} \right) \frac{\sum_{M, VV} \tau^{-1}_s}{\sum_{M, VV} + \tau^{-1}_s}
\]
\[+O \left( \left( 1 - \left( \frac{1}{R'} \right)^2 \frac{\tau^{-1}_s}{\sum_{M, VV} + \tau^{-1}_s} \right) \frac{\sum_{M, VV} \tau^{-1}_s}{\sum_{M, VV} + \tau^{-1}_s} \right)^2 \]
and substituting it into our expression for excess volatility that
\[
\left( \varphi - \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \right)' \Omega^M \left( \varphi - \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \right) - p_V \sigma_V^2 - \tilde{p}_N \sigma_N^2
\]
\[\frac{1}{2} \frac{R'}{\gamma \sigma_N} \left( 1 - \left( \frac{1}{R'} \right)^2 \frac{\tau^{-1}_s}{\sum_{M, VV} + \tau^{-1}_s} \right) - 1\]
\[\frac{1}{2} \frac{R'}{\gamma \sigma_N} \frac{1}{4} \left( \frac{R'}{\tau^{-1}_s} \right)^2 - \sigma_D^2 - \left( \frac{\sigma_V}{R'} \right)^2 \]
\[\frac{1}{2} \frac{R'}{\gamma \sigma_N} \frac{\sum_{M, VV}}{\sum_{M, VV} + \tau^{-1}_s} \]
\[-1, \]
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from which follows that it is sufficient for the difference to be positive that

\[
\frac{\sum_{M,VV}}{\sum_{M,VV} + \tau_s^{-1}} > \sqrt{\left(\frac{1}{2} \frac{Rf}{\gamma \sigma_N}\right)^2 - \sigma_D^2 - \left(\frac{\sigma_V}{Rf}\right)^2},
\]

where \(\sqrt{\left(\frac{1}{2} \frac{Rf}{\gamma \sigma_N}\right)^2 - \sigma_D^2 - \left(\frac{1}{Rf}\right)^2 \sigma_V^2} \leq 1\) when the square root is real. Notice that the RHS is decreasing in \(\gamma \sigma_N, \sigma_D, \) and \(\sigma_V\), which relaxes the sufficient condition, and increasing in \(Rf\).

Now we recognize that \(\Sigma_{M,VV}\) is increasing in \(\sigma_N^2\). To see this, we simplify the identifying condition for \(\Sigma_{M,VV}\) to

\[
\frac{\sum_{M,VV}}{\sigma_V^2} = \left(\frac{p_N \sigma_N}{p_V \sigma_V}\right)^2 + 1,
\]

and see that \(\Sigma_{M,VV}\) is increasing in the noise-to-signal ratio, \(\left(\frac{p_N \sigma_N}{p_V \sigma_V}\right)^2\). Recognizing that

\[
\frac{\sum_{M,VV}}{\sum_{M,VV} + \tau_s^{-1}} = \left(1 + \frac{\tau_s^{-1}}{\sigma_V^2}\right) \left(\frac{p_N \sigma_N}{p_V \sigma_V}\right)^2 + \frac{\tau_s^{-1}}{\sigma_V^2}
\]

then, given the definition of \(p_N\), we can express \(\frac{p_N \sigma_N}{p_V \sigma_V}\) as

\[
\frac{p_N \sigma_N}{p_V \sigma_V} \left(\frac{\left(\frac{p_N \sigma_N}{p_V \sigma_V}\right)^2}{\left(1 + \frac{\tau_s^{-1}}{\sigma_V^2}\right) \left(\frac{p_N \sigma_N}{p_V \sigma_V}\right)^2 + \frac{\tau_s^{-1}}{\sigma_V^2}}\right)
\]

\[
= -\sqrt{\left(\frac{1}{2} \frac{Rf}{\gamma \sigma_N}\right)^2 - \sigma_D^2 - \left(\frac{\sigma_V}{Rf}\right)^2} - \left(1 - \left(\frac{1}{Rf}\right)^2 \left(1 - \frac{\sum_{M,VV}}{\sum_{M,VV} + \tau_s^{-1}}\right) \frac{\sum_{M,VV} \tau_s^{-1}}{\sum_{M,VV} + \tau_s^{-1}}\right) \frac{\sum_{M,VV}}{\sum_{M,VV} + \tau_s^{-1}} + \frac{1}{2} \frac{Rf}{\gamma \sigma_N}.
\]

Since the LHS is an increasing function of \(\frac{p_N \sigma_N}{p_V \sigma_V}\) (since \(\frac{p_N}{p_V} > 0\)), while the RHS is increasing in \(\sigma_N\) by the comparative static for \(p_N\), it follows that \(\frac{p_N \sigma_N}{p_V \sigma_V}\) is increasing in \(\sigma_N\).

Since the LHS of the sufficient condition is increasing in \(\sigma_N\) through \(\frac{\sum_{M,VV}}{\sum_{M,VV} + \tau_s^{-1}}\), while the RHS is decreasing in \(\sigma_N\), it follows that there is a critical \(\sigma_N^*\) such that price volatility with imperfect information is higher.
Similarly, one can express the deviation of the price from its fundamental value as

\[
\text{Var}[P_{t+1} - p_V V_{t+2} | \mathcal{F}_t^M] = \text{Var}[\Delta P_{t+1} | \mathcal{F}_t^M] + \left(p_V^2 - 2p_V p_{V'}\right) \sigma_V^2 - 2p_V \left(p_V - p_{V'}\right) \left(1 - \frac{\left(\frac{p_{V'} \sigma_N}{p_{V'} \sigma_V}\right)^2}{1 + \left(\frac{p_{V'} \sigma_N}{p_{V'} \sigma_V}\right)^2}\right) \sigma_{V'}^2
\]

from our definition of \(\Sigma_{M,VV}\). Subtracting its perfect information counterpart, \(p_N^2 \sigma_N^2\), we arrive at

\[
\text{Var}[P_{t+1} - p_V V_{t+2} | \mathcal{F}_t^M] - p_N^2 \sigma_N^2 = \text{Var}[\Delta P_{t+1} | \mathcal{F}_t^M] - p_V^2 \sigma_V^2 - p_N^2 \sigma_N^2 + 2p_V \left(\frac{\Sigma_{M,VV} \tau_s^{-1}}{\Sigma_{M,VV} + \tau_s^{-1}}\right)
\]

Notice when \(\text{Var}[\Delta P_{t+1} | \mathcal{F}_t^M] > p_V^2 \sigma_V^2 + p_N^2 \sigma_N^2\), or price volatility is higher than with perfect information, then it also follows that

\[
\text{Var}[P_{t+1} - p_V V_{t+2} | \mathcal{F}_t^M] > p_N^2 \sigma_N^2.
\]

**Proof of Proposition A3**

To arrive at the beliefs of investors and the government, we first characterize the market beliefs based on the public information set \(\mathcal{F}_t^M\). To derive the market beliefs, we proceed in several steps. First, we assume the market posterior belief of \((V_{t+1}, N_t, G_{t+1})\) is jointly Gaussian, \((V_{t+1}, N_t, G_{t+1}) \sim \mathcal{N}\left(\left(\hat{V}_t^M, \hat{N}_t^M, \hat{G}_t^M\right), \Sigma_t^M\right)\), where:

\[
\begin{bmatrix}
\hat{V}_t^M \\
\hat{N}_t^M \\
\hat{G}_t^M \\
G_t
\end{bmatrix}
= E
\begin{bmatrix}
V_{t+1} \\
N_t \\
G_{t+1} \\
G_t
\end{bmatrix}
| \mathcal{F}_t^M,
\]

\[
\Sigma_t^M =
\begin{bmatrix}
\Sigma_{t,M,VV} & \Sigma_{t,M,VN} & \Sigma_{t,M,VG} & 0 \\
\Sigma_{t,M,NV} & \Sigma_{t,M,NN} & \Sigma_{t,M,NG} & 0 \\
\Sigma_{t,M,GV} & \Sigma_{t,M,NV} & \Sigma_{t,M,GG} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]
Standard results for the Kalman Filter then establish that the law of motion of the conditional expectation of the market’s posterior beliefs \( (\hat{V}_t^M, \hat{N}_t^M) \) is:

\[
\begin{bmatrix}
\hat{V}_{t+1}^M \\
\hat{N}_{t+1}^M \\
\hat{\eta}_{t+1}^M \\
G_{t+1}
\end{bmatrix} = 
\begin{bmatrix}
\rho_V & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{V}_t^M \\
\hat{N}_{t-1}^M \\
\hat{\eta}_{t-1}^M \\
G_{t-1}
\end{bmatrix} +
\begin{bmatrix}
D_t - \hat{V}_t^M \\
\eta_t^M - p_V \rho_V \hat{V}_t^M \\
G_t - G_{t|t-1}
\end{bmatrix},
\]

where:

\[
K_t^M = \text{CoV} \begin{bmatrix} V_{t+1} \\ N_t \\ G_{t+1} \\ G_t \end{bmatrix},
\]

\[
\times \text{Var} \begin{bmatrix} D_t - \hat{V}_t^M \\ \eta_t^M - p_V \rho_V \hat{V}_t^M \\ G_t - G_{t|t-1} \end{bmatrix}^{-1} \times \text{CoV} \begin{bmatrix} V_{t+1} \\ N_t \\ G_{t+1} \\ G_t \end{bmatrix},
\]

is the Kalman Gain, and that the conditional variance \( \Sigma_t^M \) evolves deterministically according to:

\[
\Sigma_t^M = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} \Sigma_{t-1}^M 
\]

\[
- K_t^M \text{CoV} \begin{bmatrix} D_t - \hat{V}_t^M \\ \eta_t^M - p_V \rho_V \hat{V}_t^M \\ G_t - G_{t|t-1} \end{bmatrix},
\]

It is straightforward to compute that:

\[
\text{CoV} \begin{bmatrix} V_{t+1} \\ N_t \\ G_{t+1} \\ G_t \end{bmatrix},
\]

\[
= \begin{bmatrix}
\rho_V \Sigma_{t-1}^{M,VVV} + p_V \rho_V \Sigma_{t-1}^{M,VVG} + \sigma_V^2 \\
0 & p_N \sigma_N^2 \\
0 & p_G \sigma_G^2 \\
\Sigma_{t-1}^{M,VG} & p_V \rho_V \Sigma_{t-1}^{M,VVG} + \sigma_V^2
\end{bmatrix},
\]

\[
= \begin{bmatrix}
\rho_V \Sigma_{t-1}^{M,VVV} + p_V \rho_V \Sigma_{t-1}^{M,VVG} + \sigma_V^2 \\
0 & p_N \sigma_N^2 \\
0 & p_G \sigma_G^2 \\
\Sigma_{t-1}^{M,VG} & p_V \rho_V \Sigma_{t-1}^{M,VVG} + \sigma_V^2
\end{bmatrix},
\]

\[
= \begin{bmatrix}
\rho_V \Sigma_{t-1}^{M,VVV} + p_V \rho_V \Sigma_{t-1}^{M,VVG} + \sigma_V^2 \\
0 & p_N \sigma_N^2 \\
0 & p_G \sigma_G^2 \\
\Sigma_{t-1}^{M,VG} & p_V \rho_V \Sigma_{t-1}^{M,VVG} + \sigma_V^2
\end{bmatrix},
\]

\[
= \begin{bmatrix}
\rho_V \Sigma_{t-1}^{M,VVV} + p_V \rho_V \Sigma_{t-1}^{M,VVG} + \sigma_V^2 \\
0 & p_N \sigma_N^2 \\
0 & p_G \sigma_G^2 \\
\Sigma_{t-1}^{M,VG} & p_V \rho_V \Sigma_{t-1}^{M,VVG} + \sigma_V^2
\end{bmatrix},
\]

\[
= \begin{bmatrix}
\rho_V \Sigma_{t-1}^{M,VVV} + p_V \rho_V \Sigma_{t-1}^{M,VVG} + \sigma_V^2 \\
0 & p_N \sigma_N^2 \\
0 & p_G \sigma_G^2 \\
\Sigma_{t-1}^{M,VG} & p_V \rho_V \Sigma_{t-1}^{M,VVG} + \sigma_V^2
\end{bmatrix},
\]

\[
= \begin{bmatrix}
\rho_V \Sigma_{t-1}^{M,VVV} + p_V \rho_V \Sigma_{t-1}^{M,VVG} + \sigma_V^2 \\
0 & p_N \sigma_N^2 \\
0 & p_G \sigma_G^2 \\
\Sigma_{t-1}^{M,VG} & p_V \rho_V \Sigma_{t-1}^{M,VVG} + \sigma_V^2
\end{bmatrix},
\]
and that:

\[
\Omega^{M}_{t-1} = \text{Var} \left[ \begin{bmatrix} D_t - \hat{V}^M_t \\ \eta^M_t - p_V \rho_V \hat{V}^M_t \\ G_t - G_{t|t-1} \end{bmatrix} \bigg| \mathcal{F}^{M}_{t-1} \right]
\]

\[
= \begin{bmatrix} 
\Sigma^{M,\text{VV}}_{t-1} + \sigma_D^2 \\
\rho_V \rho_V \Sigma^{M,\text{VV}}_{t-1} p_V^2 \Sigma^{M,\text{VV}}_{t-1} + \sigma^2_V 
\end{bmatrix} \\
\rho_V \rho_V \Sigma^{M,\text{VG}}_{t-1} p_V^2 \Sigma^{M,\text{VG}}_{t-1} + \rho^2_V \sigma^2_V + \rho^2_G \sigma^2_G \\
\rho_V \rho_V \Sigma^{M,\text{VG}}_{t-1} p_V^2 \Sigma^{M,\text{VG}}_{t-1} + \rho^2_V \sigma^2_V + \rho^2_G \sigma^2_G
\end{bmatrix}.
\]

Since \( \eta^{M}_t \in \mathcal{F}^{M}_t \subseteq \mathcal{F}_t \), I can express \( \eta^{M}_t \) as:

\[
\eta^{M}_t = p_V V_{t+1} + p_N N_t + p_G G_{t+1} = p_V \hat{V}^{M}_{t+1} + p_N \hat{N}^{M}_{t+1} + p_G \hat{G}^{M}_{t+1},
\]

from which follows that:

\[
p_V \left( V_{t+1} - \hat{V}^{M}_{t+1} \right) + p_N \left( N_t - \hat{N}^{M}_{t+1} \right) + p_G \left( G_{t+1} - \hat{G}^{M}_{t+1} \right) = 0.
\]

As a consequence, it must be that the market beliefs about \( V_t \) and \( N_t \) are ex-post correlated after observing the stock price innovation process \( \eta^{M}_t \), such that we have the three identities by taking its variance and its covariance with \( V_{t+1} - \hat{V}^{M}_{t+1} \) and \( N_t - \hat{N}^{M}_{t+1} \):

\[
\Sigma^{M,\text{VN}}_{t} = -\frac{p_V}{p_N} \Sigma^{M,\text{VV}}_{t} - \frac{p_G}{p_N} \Sigma^{M,\text{VG}}_{t},
\]

\[
\Sigma^{M,\text{NN}}_{t} = -\frac{p_V}{p_N} \Sigma^{M,\text{VN}}_{t} - \frac{p_G}{p_N} \Sigma^{M,\text{NG}}_{t},
\]

\[
\Sigma^{M,\text{NG}}_{t} = -\frac{p_V}{p_N} \Sigma^{M,\text{VG}}_{t} - \frac{p_G}{p_N} \Sigma^{M,G1} \text{.}
\]

This completes our characterization of the market’s beliefs.

**Proof of Proposition A4**

Updating the market beliefs to each investor’s private beliefs can be done in a manner similar to that in He and Wang (1995). Note that the market beliefs act as the prior for investor \( i \) who observes the normally distributed private signal \( s^{i}_t \). The posterior of investor \( i \) is

\[
N \left( \begin{bmatrix} \hat{V}^{i}_{t+1}, \hat{N}^{i}_{t}, \hat{G}^{i}_{t+1|t} \end{bmatrix}, \Sigma^{i} \right), \text{ where } \begin{bmatrix} \hat{V}^{i}_{t+1}, \hat{N}^{i}_{t}, \hat{G}^{i}_{t+1|t} \end{bmatrix} = E \left[ \begin{bmatrix} V_{t+1}, N_t, G_{t+1} \end{bmatrix} | \mathcal{F}^{i}_t \right] \text{ and } \Sigma^{i}_t (i) = E \left[ \begin{bmatrix} V_{t+1} - \hat{V}^{i}_{t+1} \\
N_t - \hat{N}^{i}_{t} \\
G_{t+1} - \hat{G}^{i}_{t+1|t} \end{bmatrix} \bigg| \mathcal{F}^{i}_t \right] \text{ are given by:}
\]

\[
\begin{bmatrix}
\hat{V}^{i}_{t+1} \\
\hat{N}^{i}_{t} \\
\hat{G}^{i}_{t+1}
\end{bmatrix} = \Gamma^{i}_t \begin{bmatrix}
s^{i}_t - \hat{V}^{M}_{t+1} \\
\hat{N}^{i}_{t} - \hat{N}^{M}_{t} \\
\hat{G}^{i}_{t+1} - \hat{G}^{M}_{t+1}
\end{bmatrix},
\]

\[
\begin{bmatrix}
\hat{V}^{i}_{t+1} \\
\hat{N}^{i}_{t} \\
\hat{G}^{i}_{t+1}
\end{bmatrix} = \left[ \begin{bmatrix}
D_t - \hat{V}^{M}_t \\
\eta^M_t - p_V \rho_V \hat{V}^M_t \\
G_t - G_{t|t-1} \end{bmatrix} \bigg| \mathcal{F}^{M}_{t-1} \right]
\]

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where:

\[
\Gamma_t' = \text{CoV} \begin{bmatrix}
V_{t+1} & s_t - \hat{V}_{t+1}^M \\
N_t & g_t - G_{t+1}^M
\end{bmatrix}
\begin{bmatrix}
F_{t-1}^M \\
F_t^M
\end{bmatrix}
\begin{bmatrix}
\text{Var} \begin{bmatrix}
F_t^M \\
F_{t-1}^M
\end{bmatrix}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\Sigma_{t,M,VV} & \Sigma_{t,M,VG_1} \\
\Sigma_{t,M,VN} & \Sigma_{t,M,NG_1} \\
\Sigma_{t,M,VG_1} & \Sigma_{t,M,G_1 G_1}
\end{bmatrix}
\begin{bmatrix}
\Sigma_{t,M,VV} + (a^i \tau_s)^{-1} & \Sigma_{t,M,VG_1} \\
\Sigma_{t,M,VN} & \Sigma_{t,M,G_1 G_1} + [(1-a^i) \tau_g]^{-1}
\end{bmatrix}
\]

and:

\[
\Sigma_t^i (i) = \Sigma^M_t - \Gamma_t^i
\]

Since \( G_t \) is publicly revealed, it is common knowledge and speculators need not update their beliefs about it with their private information. This characterizes the beliefs of investors given the market’s beliefs.

### Proof of Proposition A5

After the system has run for a sufficiently long time, initial conditions will diminish and the conditional variance of the Kalman Filter for the market beliefs \( \Sigma^M_t \) will settle down to its deterministic, covariance-stationary steady-state. To see this, let us conjecture that \( \Sigma^M_t \rightarrow \Sigma^M \). In this proposed steady-state, \( \Gamma_t \rightarrow \Gamma \), where \( \Gamma \) is given by:

\[
\Gamma = \begin{bmatrix}
\Sigma_{t,M,VV} & \Sigma_{t,M,VG_1} \\
\Sigma_{t,M,VN} & \Sigma_{t,M,NG_1} \\
\Sigma_{t,M,VG_1} & \Sigma_{t,M,G_1 G_1}
\end{bmatrix}
\begin{bmatrix}
\Sigma_{t,M,VV} + (a^i \tau_s)^{-1} & \Sigma_{t,M,VG_1} \\
\Sigma_{t,M,VN} & \Sigma_{t,M,G_1 G_1} + [(1-a^i) \tau_g]^{-1}
\end{bmatrix}
\]

Consequently, since \( \Gamma \) is indeed constant, so is \( \Sigma^M_t \). Furthermore, the steady-state Kalman Gain \( K^M \) is given by:

\[
K^M = \begin{bmatrix}
\rho_V \Sigma^M_{V,V} & \rho_V \left( \rho_V^2 \Sigma^M_{V,V} + \sigma_V^2 \right) & \rho_V \Sigma^M_{V,G}
\\
0 & \rho_N \Sigma^M_{N,N} & 0
\\
0 & 0 & \rho_G \Sigma^M_{G,G}
\end{bmatrix}
\Omega^M^{-1}
\]

where:

\[
\Omega^M = \begin{bmatrix}
\Sigma^M_{V,V} + \sigma_D^2 \\
\rho_V \rho_{V'} \Sigma^M_{V,V} \\
\rho_V \rho_{V'} \Sigma^M_{V,V}
\end{bmatrix}
\begin{bmatrix}
\rho_V \rho_{V'} \Sigma^M_{V,V} \\
\rho_V \rho_{V'} \Sigma^M_{V,V} \\
\rho_V \rho_{V'} \Sigma^M_{V,V}
\end{bmatrix}
\]

Consequently, since we have constructed a steady-state for the Kalman Filter for the market beliefs, such a steady-state exists.
Proof of Proposition A6

Similar to the problem for the government, it is convenient to define the state vector \( \Psi_t = [\hat{V}^{M}_{t+1}, \hat{N}^{M}_{t}, \hat{G}^{M}_{t+1}, G_t] \) with law of motion:

\[
\Psi_{t+1} = \begin{bmatrix} \rho_V & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Psi_t + K^{M}_{t} \varepsilon^{M}_{t+1},
\]

and \( \varepsilon^{M}_{t+1} \mid \mathcal{F}^{M}_t \sim N(0_{3 \times 1}, \Omega^{M}) \) is given by:

\[
\varepsilon^{M}_{t+1} = \begin{bmatrix} D_{t+1} - \hat{V}^{M}_{t+1} \\ \eta^{M}_{t+1} - \rho_V \rho_V \hat{V}^{M}_{t+1} \\ G_{t+1} - \hat{G}^{M}_{t+1} \end{bmatrix},
\]

with \( \Omega^{M} \) given in the proof of Corollary 1.

Given that excess payoffs are normally distributed, we can decompose \( R_{t+1} \) as:

\[
R_{t+1} = E[R_{t+1} \mid \mathcal{F}^{i}_t] + \phi' \varepsilon^{S,i}_{t+1}
\]

\[
= \varsigma \Psi_t + \phi' \omega \left[ \begin{array}{c} \sum^{M,VV} + (a^i \tau_s)^{-1} \\ \sum^{M, VG} \\ \sum^{M, G1} + [(1 - a^i) \tau_g]^{-1} \\ \sum^{M, VG} + (a^i \tau_s)^{-1} \end{array} \right]^{-1} \left[ \begin{array}{c} s^i_t - \hat{V}^{M}_{t+1} \\ g^i_t - \hat{G}^{M}_{t+1} \end{array} \right] + \phi' \varepsilon^{S,i}_{t+1},
\]

where:

\[
\varepsilon^{S,i}_{t+1} = \begin{bmatrix} D_{t+1} - \hat{V}^{i}_{t+1} \\ \eta^{i}_{t+1} - \rho_V \rho_V \hat{V}^{i}_{t+1} \\ G_{t+1} - \hat{G}^{i}_{t+1} \end{bmatrix},
\]

and:

\[
\varsigma = \left[ 1 + p_V (\rho_V - R^f) - p_N R^f R^g - R^f p_g \right],
\]

\[
\phi = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} p_V - p_V \\ 0 \end{bmatrix} + \begin{bmatrix} p_G - p_G \\ p_g \end{bmatrix}.
\]

In this decomposition, we have updated the investor’s beliefs sequentially from the market
beliefs following Bayes’ Rule as:

\[
E \left[ R_{t+1} \mid F_t \right] \\
= E \left[ R_{t+1} \mid F_t^M \right] + \phi' \omega \left[ \frac{\Sigma^{M,VV} + (a^i \tau_s)^{-1}}{\Sigma^{M,GG}_i} \Sigma^{M,GG}_i + [(1 - a^i) \tau_g]^{-1} \right] \left[ s^i_t - \hat{V}^M_{t+1} \right] \left[ g^i_t - G^M_{t+1} \right]^{-1} \\
= \epsilon \Psi_t + \frac{\phi' \omega}{(\Sigma^{M,VV} + (a^i \tau_s)^{-1}) (\Sigma^{M,GG}_i + [(1 - a^i) \tau_g]^{-1} - (\Sigma^{M,GG}_i)^2),}
\]

where, as in Proposition 4:

\[
\omega = \text{CoV} \left[ \frac{s^i_t - \hat{V}^M_{t+1}}{\epsilon \Psi_t + \frac{\phi' \omega}{(\Sigma^{M,VV} + (a^i \tau_s)^{-1}) (\Sigma^{M,GG}_i + [(1 - a^i) \tau_g]^{-1} - (\Sigma^{M,GG}_i)^2),}} \right] \\
= \begin{bmatrix}
\Sigma^{M,GG}_i + [(1 - a^i) \tau_g]^{-1} - \Sigma^{M,GG}_i \\
-\Sigma^{M,GG}_i & \Sigma^{M,GG}_i + [(1 - a^i) \tau_g]^{-1} - (\Sigma^{M,GG}_i)^2
\end{bmatrix}
\end{bmatrix}.
\]

Similarly, by Bayes’ Rule, \( \epsilon^S_{t+1} \mid F_t^i \sim N (0_{2 \times 1}, \Omega^S) \), where:

\[
\Omega^S = \Omega^M - \frac{\omega}{(\Sigma^{M,GG}_i + [(1 - a^i) \tau_g]^{-1} - (\Sigma^{M,GG}_i)^2)} \omega'
\]

Standard results establish that the investor’s problem is equivalent to the mean-variance optimization program:

\[
\sup_{X_i(i)} \left\{ R^i W + X_i^i E \left[ R_{t+1} \mid F_t^i \right] - \frac{\gamma}{2} X_i^i \text{Var} \left[ R_{t+1} \mid F_t^i \right] \right\}
\]

Importantly, since the investors have to form conditional expectations about excess payoffs at \( t + 1 \), they must form conditional expectations about the government’s future trading \( E \left[ G_{t+1} \mid F_t^i \right] \). Given that the investors are price-takers, from the FOC we see that the optimal investment of investor \( i \) in the risky asset is given by:

\[
X^i_t = \frac{E \left[ R_{t+1} \mid F_t^i \right]}{\gamma \text{Var} \left[ R_{t+1} \mid F_t^i \right]}
\]

\[
= \frac{\phi' \omega \left[ \frac{\Sigma^{M,GG}_i + [(1 - a^i) \tau_g]^{-1}}{\Sigma^{M,GG}_i + [(1 - a^i) \tau_g]^{-1} - (\Sigma^{M,GG}_i)^2}} \right] \left[ s^i_t - \hat{V}^M_{t+1} \right] \left[ g^i_t - G^M_{t+1} \right]^{-1} \\
= \frac{1}{\gamma} \epsilon \Psi_t + \frac{\phi' \omega \left[ \frac{\Sigma^{M,GG}_i + [(1 - a^i) \tau_g]^{-1}}{\Sigma^{M,GG}_i + [(1 - a^i) \tau_g]^{-1} - (\Sigma^{M,GG}_i)^2}} \right] \left[ s^i_t - \hat{V}^M_{t+1} \right] \left[ g^i_t - G^M_{t+1} \right]^{-1} \\
= \frac{\phi' \Omega^M \phi}{(\Sigma^{M,GG}_i + [(1 - a^i) \tau_g]^{-1} - (\Sigma^{M,GG}_i)^2)}
\]

This completes our characterization of the optimal trading policy of the investors.
Proof of Proposition A7

Each investor faces the optimization problem (A1) given in the main paper. It then follows that investor $i$ will choose to learn about the payoff fundamental $V_t$ (i.e., $a^i_t = 1$) with probability $\lambda$:

$$
\lambda = \left\{ \begin{array}{ll}
1, & Q < 0 \\
(0, 1), & Q = 0 \\
0, & Q > 0,
\end{array} \right.
$$

where:

$$
Q = \phi'(M(0) - M(1)) \phi = \phi' \omega \left[ \begin{array}{ccc}
\frac{1}{\sum M_{VV} + \tau_s^{-1}} & 0 \\
0 & \frac{1}{\sum M_{G1} + \tau_g^{-1}}
\end{array} \right] \omega' \phi.
$$

Given $\omega$, we can expand out this condition to arrive at:

$$
Q = \left( 1 + (p_V - p_V) K_{1,1}^M + (p_G - p_G) K_{3,1}^M \right) \left( 1 + (p_V - p_V) K_{1,2}^M + (p_G - p_G) K_{3,2}^M \right)
$$

$$
= \frac{\left( 1 + (p_V - p_V) K_{1,1}^M + (p_G - p_G) K_{3,1}^M \right) \left( 1 + (p_V - p_V) K_{1,2}^M + (p_G - p_G) K_{3,2}^M \right)}{\sum M_{G1G1} + \tau_g^{-1}}
$$

$$
- \frac{\left( 1 + (p_V - p_V) K_{1,1}^M + (p_G - p_G) K_{3,1}^M \right) \left( 1 + (p_V - p_V) K_{1,2}^M + (p_G - p_G) K_{3,2}^M \right)}{\sum M_{VV} + \tau_s^{-1}}
$$

Recognizing that $\phi' \omega = \text{Cov} \left[ \begin{array}{c} R_{t+1} \\
V_{t+1} \\
G_{t+1}
\end{array} \right] | F_t^M$, we can rewrite the above more generally as:

$$
Q = \frac{\text{Cov} \left[ \begin{array}{c} R_{t+1}, V_{t+1} \\
G_{t+1}
\end{array} \right] | F_t^M}{\sum M_{G1G1} + \tau_g^{-1}} - \frac{\text{Cov} \left[ \begin{array}{c} R_{t+1}, V_{t+1} \\
G_{t+1}
\end{array} \right] | F_t^M}{\sum M_{VV} + \tau_s^{-1}}.
$$

Proof of Proposition 5

In the special case that $p_V = 0$, it follows that the Kalman Gain, the steady-state market beliefs, and the $Q$–statistic for information acquisition satisfy:

$$
K^M = \begin{bmatrix}
\frac{p_V \sigma_V^2}{\rho_G \sigma_V^2 + \rho_N \sigma_N^2 + \rho_G \sigma_G^2} & 0 \\
\frac{p_N \sigma_N^2}{\rho_G \sigma_V^2 + \rho_N \sigma_N^2 + \rho_G \sigma_G^2} & 0 \\
\frac{p_G \sigma_G^2}{\rho_G \sigma_V^2 + \rho_N \sigma_N^2 + \rho_G \sigma_G^2} & 0 \\
0 & 0
\end{bmatrix},
$$

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and:

\[
\Sigma^M = \begin{bmatrix}
\frac{p^2_N}{p^2_N + p^2_G} \sigma_N^2 + \frac{1}{p^2_N + p^2_G} \sigma_G^2 & 
\frac{p^2_N}{p^2_N + p^2_G} \sigma_N^2 + \frac{1}{p^2_N + p^2_G} \sigma_G^2 & 
\frac{p^2_N}{p^2_N + p^2_G} \sigma_N^2 + \frac{1}{p^2_N + p^2_G} \sigma_G^2 & 
\frac{p^2_N}{p^2_N + p^2_G} \sigma_N^2 + \frac{1}{p^2_N + p^2_G} \sigma_G^2 & 
0
\end{bmatrix},
\]

and:

\[
Q = \frac{p^2_N \sigma_N^2 + p^2_G \sigma_G^2 + \frac{1}{R^2} p^2_N \sigma_N^2 - p_N p_G \sigma_G^2}{p^2_N \sigma_N^2 + p^2_G \sigma_G^2} p^2_g \left( p^2_N + p^2_N \sigma_N^2 \right) \left( p^2_N \sigma_N^2 + p^2_G \sigma_G^2 \right)^2 \left( \frac{\sigma_G^2}{\sigma_N^2 + \sigma_G^2} \right)^2,
\]

respectively.

In a government-centric equilibrium, \( p_V = 0 \), and, from the market-clearing conditions, \( p_g \) and \( p_G \) satisfy:

\[
p_g = \frac{p_N \sigma_N}{1 - \theta_N} \sqrt{\frac{p^2_N \sigma_N^2}{p^2_N \sigma_N^2 + p^2_G \sigma_G^2}} \theta_N^2,
\]

\[
p_G = \frac{1}{R^2} \left( 1 - \theta_N \right) \frac{p_G p^2_N \sigma_N^2}{p^2_N \sigma_N^2 + p^2_G \sigma_G^2} \theta_N^2 \sigma_G^2,
\]

from which follows that \( p_G \) is given by, \( p_G^2 = x p^2_N \sigma_N^2 \) where \( x \) satisfies:

\[
x (1 + x)^3 = \left( \frac{\theta_N}{R^2} \sigma_G^3 \right)^2,
\]

where \( x \) is increasing in \( \frac{\theta_N}{R^2} \sigma_G^3 \). It then follows that \( Q \) reduces to:

\[
Q = \left( \frac{\sigma_G^2 - R^2}{\sigma_G^2 + (1 + x) \tau_s^2} \right)^2 \left( \frac{\theta_N}{1 - \theta_N} \right)^2 \frac{p^2_N \sigma_N^2}{\sigma_N^2 + \sigma_G^2} \left( \frac{\theta_N}{1 - \theta_N} \right)^2 \left( \frac{1 + x}{\theta_N} \right)^2 \left( \frac{\sigma_G^2}{1 + x} \right)^2
\]

which suggests that, for \( Q \geq 0 \), it must be the case that:

\[
p^2_N > \tilde{p}^2_N = \frac{\sigma_G^4}{\sigma_N^4} \left( \frac{(1 + x) \tau_s^2}{\sigma_G^2 + \sigma_G^2} \right)^2 \left( \frac{1 + x}{\theta_N} \right)^2 \left( \frac{1 + x}{\theta_N} \right)^2.
\]
Furthermore, it is straightforward to compute that:

$$\phi' M \phi = \sigma_V^2 + \sigma_D^2 + \sigma_G^2 \left( \frac{1}{1 + x} \frac{\vartheta_N}{1 - \vartheta_N} \right)^2 p_N^2 \sigma_N^2 + \frac{\left( 1 + \frac{1 + x}{1 - \vartheta_N} \frac{1}{\sigma_G^2} \right)^2}{1 + x} p_N^2 \sigma_N^2,$$

and therefore, from market-clearing, that $p_N$ also satisfies:

$$0 = \left( \sigma_G^2 \sigma_G^2 + 2 \left( 1 + x \right) \tau^{-1}_g \left( \frac{1}{1 + x} \frac{\vartheta_N}{1 - \vartheta_N} \right)^2 + \frac{\left( 1 + \frac{1 + x}{1 - \vartheta_N} \frac{1}{\sigma_G^2} \right)^2}{1 + x} \right) \sigma_N^2 p_N^2 - \frac{R^f}{1 - \vartheta_N} \sigma_G^2 \left( 1 + x \right) \tau^{-1}_g p_N + \sigma_V^2 + \sigma_D^2.$$

It follows that $p_N$ is given by the two roots of the above quadratic form:

$$p_N = \frac{1}{2 \sigma_N c} R^f \pm \sqrt{\left( \frac{1}{2 \sigma_N c} \frac{R^f}{1 - \vartheta_N} \right)^2 - \frac{\sigma_V^2 + \sigma_D^2}{\sigma_N^2}}$$

where:

$$c = \sigma_G^2 \sigma_G^2 + 2 \left( 1 + x \right) \tau^{-1}_g \left( \frac{1}{1 + x} \frac{\vartheta_N}{1 - \vartheta_N} \right)^2 + \frac{\left( 1 + \frac{1 + x}{1 - \vartheta_N} \frac{1}{\sigma_G^2} \right)^2}{1 + x} \geq 0,$$

and $c = c \left( \vartheta_N, R^f, \sigma_G \right)$. When $p_N$ exists, one consequently has that $p_N > 0$. Selecting the less positive root, and recognizing that $Q \geq 0$ whenever $p_N \geq \tilde{p}_N$, we can express this condition as:

$$\frac{\sqrt{\sigma_V^2 + \tau^{-1}_g}}{\sigma_V^2} \left( \frac{1}{2 \sigma_N c} \frac{R^f}{1 - \vartheta_N} - \sqrt{\left( \frac{1}{2 \sigma_N c} \frac{R^f}{1 - \vartheta_N} \right)^2 - \frac{\sigma_V^2 + \sigma_D^2}{\sigma_N^2}} \right) \geq (1 + x) \sqrt{\left( \sigma_G^2 + (1 + x) \tau^{-1}_g \right) \left( \frac{1 - \vartheta_N}{\sigma_G^2 - R^f \frac{x}{1 - \vartheta_N}} \right)^2}.$$

Notice that the LHS of equation (1) is always nonnegative, since it is $\frac{\sqrt{\sigma_V^2 + \tau^{-1}_g}}{\sigma_V^2} p_N \sigma_N$, and that $c$ and the RHS of equation (1) is independent of $\{\sigma_N, \sigma_V, \sigma_D\}$ since $x = x \left( \vartheta_N, R^f, \sigma_G \right)$.

Since it is straightforward to compute that:

$$\frac{dp_N \sigma_N}{d \sigma_N} = -\frac{1}{\sigma_N} \frac{1}{2 \sigma_N c} \frac{R^f}{1 - \vartheta_N} \sqrt{\left( \frac{1}{2 \sigma_N c} \frac{R^f}{1 - \vartheta_N} \right)^2 - \frac{\sigma_V^2 + \sigma_D^2}{\sigma_N^2}} - \frac{1}{2 \sigma_N c} \frac{R^f}{1 - \vartheta_N} > 0,$$

$$\frac{dp_N \sigma_N}{d \sigma_D} = \frac{\sigma_D}{c \sqrt{\left( \frac{1}{2 \sigma_N c} \frac{R^f}{1 - \vartheta_N} \right)^2 - \frac{\sigma_V^2 + \sigma_D^2}{\sigma_N^2}}} > 0,$$
it follows that the LHS is increasing in $\sigma_N$ and $\sigma_D$. Consequently, the existence condition for a government-centric equilibrium relaxes as $\sigma_N$ and $\sigma_D$ increase, and therefore a government-centric equilibrium is more likely to exist the higher are $\sigma_N$ and $\sigma_D$.

Finally, with respect to $\sigma_V$, we recognize that, as $\sigma_V \rightarrow 0$, $\frac{\sqrt{\sigma_V^2 + \sigma_D^2}}{\sigma_V^2}p_N\sigma_N \rightarrow \infty$, and consequently the LHS exceeds the RHS and $Q > 0$. Since $\frac{\sqrt{\sigma_V^2 + \sigma_D^2}}{\sigma_V^2}p_N\sigma_N$ is continuous in $V$, it follows that a government-centric equilibrium exists within a neighborhood of $\sigma_V = 0$, and consequently exists for $\sigma_V$ sufficiently small.

**Proof of Proposition 6**

We can express the conditional uncertainty about the deviation in the asset price from its fundamentals as:

$$
F = Var \left[ P_{t+1} - p_V V_{t+2} \mid \mathcal{F}_t^M \right]
$$

$$
= Var \left[ \left( \phi - \frac{1}{0} \right) \varepsilon_{t+1}^M - p_V \left( V_{t+2} - \rho_V \hat{V}_{t+1}^M \right) \mid \mathcal{F}_t^M \right]
$$

$$
= \left( \phi - \frac{1}{0} \right)' \Omega^M \left( \phi - \frac{1}{0} \right) + p_V^2 \left( \rho_V \Sigma^{M,\hat{V}V} + \sigma_V^2 \right)
$$

$$
-2p_V \left( \phi - \frac{1}{0} \right)' \left[ \rho_V \Sigma^{M,\hat{V}V} \rho_V \Sigma^{M,VG_1} \right],
$$

which we can rewrite as:

$$
Var \left[ P_{t+1} - p_V V_{t+2} \mid \mathcal{F}_t^M \right] = Var \left[ P_{t+1} \mid \mathcal{F}_t^M \right] + p_V^2 \left( \rho_V^2 \Sigma^{M,\hat{V}V} + \sigma_V^2 \right)
$$

$$
-2p_V \left( \phi - \frac{1}{0} \right)' \left[ \rho_V \Sigma^{M,\hat{V}V} \rho_V \Sigma^{M,VG_1} \right].
$$

In a government-centric equilibrium, $p_V = 0$ and there is no learning from prices about $V_{t+1}$, so $\Sigma^{M,\hat{V}V}$ is exogenous to government intervention. Consequently, the above reduces to

$$
Var \left[ P_{t+1} - p_V V_{t+2} \mid \mathcal{F}_t^M \right] = Var \left[ P_{t+1} \mid \mathcal{F}_t^M \right] + p_V^2 \left( \rho_V^2 \Sigma^{M,\hat{V}V} + \sigma_V^2 \right) - 2 \frac{(p_V \rho_V \Sigma^{M,\hat{V}V})^2}{\Sigma^{M,\hat{V}V} + \sigma_V^2},
$$

and minimizing price deviation, is then equivalent to minimizing $Var \left[ P_{t+1} \mid \mathcal{F}_t^M \right]$, or price volatility.