China’s Model of Managing the Financial System

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Internet Appendix

This internet appendix present proofs of the propositions in the main paper.

Proof of Proposition A1

We derive the perfect information equilibrium with trading by the government. We first conjecture that, when \( v_{t+1} \) and \( N_t \) are observable to the government and investors, the stock price takes the linear form:

\[
P_t = p_v v_{t+1} + p_N N_t + p_g G_t.
\]

Given that dividends are \( D_t = v_t + \sigma_D \varepsilon_t^D \), the stock price must react to a deterministic unit shift in \( v_{t+1} \) by the present value of dividends deriving from that shock, \( \frac{1}{R_f - \rho_v} \), it follows that \( p_v = \frac{1}{R_f - \rho_v} \). The innovations to \( v_{t+1} \) and \( N_t \) are the only source of risk and, from the perspective of all economic agents, the conditional expectation and variance of \( R_{t+1} \) are:

\[
E[R_{t+1} | F_t] = -p_N R_f N_t - R_f p_g G_t,
\]

\[
Var[R_{t+1} | F_t] = \sigma_D^2 + \left( \frac{1}{R_f - \rho_v} \right)^2 \sigma_v^2 + p_N^2 \sigma_N^2 + p_g^2 \sigma_G^2.
\]

Since all investors are identical when \( v_t \) and \( N_t \) are observable, it follows that in the CARA-Normal environment all investors have an identical mean-variance demand for the risky asset:

\[
X^S_t = \frac{1}{\gamma} E[R_{t+1} | F_t] = \frac{1}{\gamma} \frac{-p_N R_f N_t - R_f p_g G_t}{\sigma_D^2 + \left( \frac{1}{R_f - \rho_v} \right)^2 \sigma_v^2 + p_N^2 \sigma_N^2 + p_g^2 \sigma_G^2}.
\]

In the government’s intervention rule:

\[
X^G_t = \vartheta_N N_t + \vartheta_N \sigma_N G_t,
\]

\( \vartheta_N \) is determined by:

\[
U^G = \sup_{\vartheta} \left( \gamma_\sigma + \gamma_\vartheta \right) \left( \sigma_D^2 + \left( \frac{1}{R_f - \rho_v} \right)^2 \sigma_v^2 + p_N^2 \sigma_N^2 + p_g^2 \sigma_G^2 \right),
\]

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Finally, by imposing market-clearing, we arrive at:

\[ N = \frac{1}{\gamma \sigma_D^2 + \left(\frac{1}{R^f - \rho} \right)^2 \sigma_v^2 + p_N^2 \sigma_N^2 + p_g^2 \sigma_G^2} - p_N R^f N + \vartheta_N N, \]

\[ \partial_N \sigma_N^2 G_t = \frac{1}{\gamma \sigma_D^2 + \left(\frac{1}{R^f - \rho} \right)^2 \sigma_v^2 + p_N^2 \sigma_N^2 + p_g^2 \sigma_G^2} \frac{R^f p_g}{G_t} G_t, \]

which, by matching coefficients, reveals that:

\[-\frac{1}{\gamma} \frac{p_N R^f}{\sigma_D^2 + \left(\frac{1}{R^f - \rho} \right)^2 \sigma_v^2 + p_N^2 \sigma_N^2 + p_g^2 \sigma_G^2} + \vartheta_N = 1, \]

\[ p_N \frac{\vartheta_N}{1 - \vartheta_N} \sigma_N = p_g. \]

This confirms the conjectured equilibrium.

Rearranging this equation for \( p_N \), and substituting for \( p_g \), we arrive at the quadratic equation for \( p_N \):

\[ \left( 1 + \left( \frac{\vartheta_N}{1 - \vartheta_N} \right)^2 \sigma_G^2 \right) p_N^2 + \frac{R^f}{\gamma \sigma_N^2 (1 - \vartheta_N)} \rho_N + \frac{\sigma_D^2}{\sigma_N^2} \left( \frac{1}{R^f - \rho} \right)^2 \sigma_v^2 = 0, \]

from which follows that \( p_N \) has two roots:

\[ p_N (\vartheta_N) = \frac{-\frac{R^f}{\gamma \sigma_N^2 (1 - \vartheta_N)} \pm \sqrt{\left( \frac{R^f}{\gamma \sigma_N^2 (1 - \vartheta_N)} \right)^2 - 4 \left( 1 + \left( \frac{\vartheta_N}{1 - \vartheta_N} \right)^2 \sigma_G^2 \right) \left( \frac{\sigma_D^2}{\sigma_N^2} \left( \frac{1}{R^f - \rho} \right)^2 \sigma_v^2 \right)}}{2 \left( 1 + \left( \frac{\vartheta_N}{1 - \vartheta_N} \right)^2 \sigma_G^2 \right)}. \]

Recognizing that two negative solutions for \( P_N \) exist if the expression under the square root is nonnegative, it follows that the market breaks down occurs whenever:

\[ R^f < 2 (1 - \vartheta_N) \gamma \sqrt{\left( 1 + \left( \frac{\vartheta_N}{1 - \vartheta_N} \right)^2 \sigma_G^2 \right) \left( \sigma_D^2 \sigma_N^2 + \left( \frac{1}{R^f - \rho} \right)^2 \sigma_v^2 \sigma_N^2 \right)}. \]

Given that \( Var(\Delta P_t | \mathcal{F}_{t-1}) = \left( \frac{1}{R^f - \rho} \right)^2 \sigma_v^2 + p_N^2 \sigma_N^2 + p_g^2 \sigma_G^2 \), substituting for \( p_g \), the government’s optimization problem consequently reduces to:

\[ U^G = \sup_{\vartheta_N} \left( (\gamma_\sigma + \gamma_\vartheta) \left( 1 + \left( \frac{\vartheta_N}{1 - \vartheta_N} \right)^2 \sigma_G^2 \right) p_N^2 \sigma_N^2 \right). \]
and, from the two market-clearing condition restrictions on the coefficients $p_N$ and $p_g$, that

$$
\vartheta_N = 1 + \frac{1}{\gamma} \left( \frac{p_N R_f}{\sigma_D^2} + \left( \frac{1}{R_f - \rho_v} \right)^2 \sigma_u^2 + \left( 1 + \left( \frac{\vartheta_N}{1 - \vartheta_N} \right)^2 \sigma_G^2 \right) p_N^2 \sigma_N^2 \right).
$$

To establish that the linear equilibrium is the unique, symmetric equilibrium, we express each investor’s optimization problem as:

$$
U_t = \max_{X_t} E \left[ e^{-\gamma(R\tilde{W} + X_t(\nu_{t+1} + \sigma_D \xi_{t+1}^\mu + P_{t+1} - R P_t))} \mid \mathcal{F}_t \right]
$$

For an arbitrary price function $P_t$, the FOC for the investor’s holding of the risky asset $X_t$ is:

$$
E \left[ (\nu_{t+1} + \sigma_D \xi_{t+1}^\mu + P_{t+1} - R P_t) e^{-\gamma X_t(\nu_{t+1} + \sigma_D \xi_{t+1}^\mu + P_{t+1} - R P_t)} \mid \mathcal{F}_t \right] = 0.
$$

Substituting this with the market-clearing condition:

$$
X_t = (1 - \vartheta_N) N_t - \vartheta_N \sigma_N G_t,
$$

we arrive at:

$$
E \left[ (\nu_{t+1} + \sigma_D \xi_{t+1}^\mu + P_{t+1} - R P_t) e^{-\gamma((1 - \vartheta_N) N_t - \vartheta_N \sigma_N G_t)(\nu_{t+1} + \sigma_D \xi_{t+1}^\mu + P_{t+1} - R P_t)} \mid \mathcal{F}_t \right] = 0.
$$

Since $P_{t+1}$ cannot be a function of $\xi_{t+1}^\mu$, as $P_{t+1}$ is forward-looking for the new generation of investors at time $t + 1$, the above can be rewritten as:

$$
P_t = \frac{1}{R_f} \nu_{t+1} - \frac{\gamma}{R_f} \sigma_D^2 ((1 - \vartheta_N) N_t + \vartheta_N \sigma_N G_t) \quad \text{(IA.2)}
$$

$$
+ \frac{1}{R_f} E \left[ P_{t+1} \frac{e^{-\gamma((1 - \vartheta_N) N_t - \vartheta_N \sigma_N G_t) P_{t+1}}}{e^{-\gamma((1 - \vartheta_N) N_t - \vartheta_N \sigma_N G_t) P_{t+1}}} \mid \mathcal{F}_t \right].
$$

This defines a functional equation, whose fixed point is the price functional $P_t$. To see that the linear equilibrium we derived above solves this functional equation, we rewrite equation (IA.2) as:

$$
P_t = \frac{1}{R_f} \nu_{t+1} - \frac{\gamma}{R_f} \sigma_D^2 ((1 - \vartheta_N) N_t - \vartheta_N \sigma_N G_t) + \frac{1}{R_f} \left[ \partial_u \log E \left[ e^{u P_{t+1}} \mid \mathcal{F}_t \right] \right]_{u = -\gamma((1 - \vartheta_N) N_t + \vartheta_N \sigma_N G_t)},
$$

and conjecture that $P_t = \frac{1}{R_f - \rho_v} \nu_{t+1} + p_N N_t + p_g G_t$, from which follows that $p_N$ satisfies equation (IA.1). This verifies that the linear price equilibrium satisfies this more general equilibrium condition.
Now define the operator $T : \mathcal{B}(\mathbb{R}^3) \rightarrow \mathcal{B}(\mathbb{R}^3)$:

$$Tf (v, N, G) = \frac{1}{R^f} v - \frac{\gamma}{R^f} \sigma_D^2 \left((1 - \vartheta_N) N - \vartheta_N \sigma_N G\right) + \frac{1}{R^f} E \left[ \frac{f (v', N', G') e^{-\gamma((1 - \vartheta_N) N - \vartheta_N \sigma_N G) f'}}{E [e^{-\gamma((1 - \vartheta_N) N - \vartheta_N \sigma_N G) f']} | v, N, G] | v, N, G \right],$$

for $f \in \mathcal{B}(\mathbb{R}^3)$, where $\mathcal{B}(\mathbb{R}^3)$ is the space of continuous functions $\psi-$bounded in the $\psi-$norm $\|f\|_\psi = \sup_x \frac{|f(x)|}{\psi(x)}$, where $\psi (x) = 1 + \|x\|^b$ has polynomial growth for some $b \geq 1$, and $x = [v \ N \ G]$. Since $v_{t+1}, G_t$, and $N_t$ are Markov processes, $\{v_{t+1}, N_t, G_t\}$ are sufficient statistics for the conditioning in the above conditional expectations.

We now consider a sequence of increasing finite horizon stopping dates for the economy, with stopping dates $\{\tau_k\}_{k=1}^\infty$ increasing to $\infty$ as $k \nearrow \infty$, to establish uniqueness of the linear equilibrium, and then take the limit as $k \nearrow \infty$.\footnote{This allows us to avoid the issue of the second, unstable linear equilibrium that arises with infinite horizon economies because of self-fulfilling rational expectations.} At the stopping date, $P (v_{\tau_k}, N_{\tau_k}, G_{\tau_k}) = 0$ by no arbitrage. We next establish that $T$ is a contraction map by verifying that $T$ satisfies Blackwell’s Sufficiency conditions. First, to see that $Tf \in \mathcal{B}(\mathbb{R}^3)$, notice that $Tf$ continuous in $(v, N, G)$ is straightforward from application of the expectation operator to continuous functions, since the joint distribution of $(v, N, G)$ satisfies the Feller condition, and that we can bound the above as:

$$Tf (v, N, G) \leq \frac{1}{R^f} v - \frac{\gamma}{R^f} \sigma_D^2 \left((1 - \vartheta_N) N - \vartheta_N \sigma_N G\right) + \frac{1}{R^f} \sup_{(v', N', G')} f (v', N', G').$$

Since $P_t$ defines an asset price, it must be the case that, if $P (v_{t+1}, N_{t+1}, G_{t+1})$ (weakly) increases $\forall \{v_{t+2}, G_{t+1}, N_{t+1}\}$, which is a First-Order Stochastic Dominance (FOSD) shift in the distribution of $P (v_{t+1}, N_{t+1}, G_{t+1})$, then it is (weakly) preferred by any agent whose utility is increasing in wealth. Consequently, investors would demand more to earn the higher return, which would bid up the price today. Thus, $Tf \geq Tg$ for $f \geq g$, and $T$ satisfies monotonicity. Furthermore:

$$T \left( f (v, N, G) + c \psi (v, N, G) \right) = \frac{1}{R^f} v - \frac{\gamma}{R^f} \sigma_D^2 \left((1 - \vartheta_N) N - \vartheta_N \sigma_N G\right) + \frac{1}{R^f} E \left[ \frac{f (v', N', G') e^{-\gamma((1 - \vartheta_N) N - \vartheta_N \sigma_N G) f'}}{E [e^{-\gamma((1 - \vartheta_N) N - \vartheta_N \sigma_N G) f']} | v, N, G] | v, N, G \right] + \frac{1}{R^f} \psi (v, N, G)$$

and $T$ satisfies discounting since $R^f > 1$. Therefore, $T$ is a strict contraction map by the Weighted Contraction Mapping Theorem of Boyd (1990). Since a contraction map has, at
most, one fixed point, if an equilibrium with a continuous, linear $$P(v_{t+1}, N_{t+1}, G_{t+1})$$ exists, it must be the unique equilibrium in the economy, at least within the class of functions bounded in the $$\psi - \text{sup}$$ norm. Taking the limit as $$k \to \infty$$, with $$P(v_{\tau_k}, N_{\tau_k}, G_{\tau_k}) = 0$$ along the sequence acting as a transversality condition, establishes the linear equilibrium as the unique equilibrium of this limiting sequence.

**Proof of Proposition 1**

Note from the variance of the excess asset payoff that:

$$\text{Var} [R_{t+1} \mid \mathcal{F}_t] = \sigma_D^2 + \left( \frac{1}{R_f - \rho_v} \right)^2 \sigma_v^2 + p_N^2 \sigma_N^2,$$

and thus the excess volatility is driven by the $$p_N^2 \sigma_N^2$$ term. Consider now the expression for the less negative root of $$p_N$$ from Proposition 1 in the absence of government intervention:

$$p_N = -\frac{1}{2\sigma_N^2} A + \sqrt{\left( \frac{1}{2\sigma_N^2} A \right)^2 - \frac{1}{\sigma_N^2} B},$$

where:

$$A = \frac{R_f}{\gamma},$$

$$B = \sigma_D^2 + \left( \frac{1}{R_f - \rho_v} \right)^2 \sigma_v^2,$$

to simplify notation. Given this expression, it follows that:

$$p_N^2 \sigma_N^2 = \frac{A^2}{2\sigma_N^2} - A \sqrt{\left( \frac{A}{2\sigma_N^2} \right)^2 - \frac{1}{\sigma_N^2} B} - B.$$

Differentiating with respect to $$\sigma_N^2$$, we find with some manipulation that:

$$\frac{\partial p_N^2 \sigma_N^2}{\partial \sigma_N^2} = \frac{A}{2\sigma_N^2} \sqrt{\left( \frac{A}{2\sigma_N^2} \right)^2 - \frac{1}{\sigma_N^2} B}$$

which we can factorize as:

$$\frac{\partial p_N^2 \sigma_N^2}{\partial \sigma_N^2} = \frac{A}{2\sigma_N^2} \left( \frac{A}{2\sigma_N^2} - \sqrt{\left( \frac{A}{2\sigma_N^2} \right)^2 - \frac{1}{\sigma_N^2} B} \right)^2 \geq 0,$$
and, since \( P_t = \frac{1}{R_t - \rho_v} v_{t+1} + p_N N_t \) with \( v_{t+1} \) and \( N_t \) independent of each other, this completes the proof. Therefore, volatility is highest close to market breakdown, when \( \left( \frac{R_f}{\gamma \sigma_N} \right)^2 - 4 \left( \frac{\sigma^2}{\sigma_N^2} + \left( \frac{1}{R_f - \rho_v} \right)^2 \frac{\sigma^2}{\sigma_N^2} \right) = \varepsilon \) for \( \varepsilon \) arbitrarily small. Market breakdown occurs when \( \varepsilon = 0 \) or:

\[
\sigma_N = \frac{R_f}{2\gamma \sqrt{\sigma_D^2 + \left( \frac{1}{R_f - \rho_v} \right)^2 \sigma_v^2}}.
\]

Furthermore, as \( \varepsilon \to 0 \), and \( \sigma_N \to \frac{R_f}{2\gamma \sqrt{\sigma_D^2 + \left( \frac{1}{R_f - \rho_v} \right)^2 \sigma_v^2}} \), then:

\[
p_N^2 \sigma_N^2 \to \sigma_D^2 + \left( \frac{1}{R_f - \rho_v} \right)^2 \sigma_v^2.
\]

Consequently, the maximum conditional excess payoff variance before breakdown occurs is

\[
\text{Var} \left[ R_{t+1} | F_t \right] \to 2 \left( \sigma_D^2 + \left( \frac{1}{R_f - \rho_v} \right)^2 \sigma_v^2 \right).
\]

**Proof of Proposition A2**

Given its posterior for \( v_2, v_2 | \mathcal{F}_1^M \sim \mathcal{N} \left( \hat{v}_2^M, \Sigma^{M,vv} \right) \), the firm manager’s optimal choice of capital and labor satisfy:

\[
K_1 = \frac{\alpha}{r_1} E \left[ Y_2 | \mathcal{F}_1^M \right] = \left( \frac{\alpha}{r_1} \right) \frac{1}{1 - \alpha} \frac{1}{e^{v_2 + e^{y_2} + \frac{1}{2} \frac{1 - \alpha}{r_1} (\beta^2 \Sigma^{M,vv} + \sigma_v^2)}} L_1,
\]

\[
L_1 = \frac{1 - \alpha}{w} E \left[ Y_2 | \mathcal{F}_1^M \right] = \left( \frac{1 - \alpha}{w} \right) \frac{1}{e^{v_2 + e^{y_2} + \frac{1}{2} \frac{1 - \alpha}{r_1} (\beta^2 \Sigma^{M,vv} + \sigma_v^2)}} K_1.
\]

Substituting the optimal choice of labor, it follows that the realized output \( Y_2 \) is given by:

\[
Y_2 = \left( \frac{1 - \alpha}{w} \right) e^{v_2 + e^{y_2} + \frac{1 - \alpha}{w} \frac{1}{2} \frac{1 - \alpha}{r_1} (\beta^2 \Sigma^{M,vv} + \sigma_v^2)} K_1.
\]

We now turn to the problem of investors. Since the price of the asset \( P_t \) is Gaussian, we can invert the VaR constraint to arrive at the leverage constraint:

\[
i_0 I_0 \leq \bar{v} - \Phi^{-1} (1 - \pi) \sigma_P,
\]

where \( \Phi(\cdot) \) is the CDF of the normal distribution. It also follows that:

\[
E \left[ (P_2 - i_0 I_0) 1_{\{P_2\}} \right] = \sigma_P E \left[ Z_2 1_{\{Z_2 > 0\}} \right],
\]
where $Z_2 \sim \mathcal{N}\left( \frac{\bar{v} - i_0 I_0}{\sigma_p}, 1 \right)$. By the properties of the truncated normal distribution:

$$E \left[ Z_2 1_{\{Z_2 < 0\}} \right] = E \left[ Z_2 \mid Z_2 < a \right] E \left[ 1_{\{Z_2 < 0\}} \right] = \frac{\bar{v} - i_0 I_0}{\sigma_p} \Phi \left( -\frac{-\bar{v} + i_0 I_0}{\sigma_p} \right) - \phi \left( -\frac{-\bar{v} + i_0 I_0}{\sigma_p} \right),$$

$$E \left[ Z_2 1_{\{Z_2 \geq 0\}} \right] = E \left[ Z_2 \right] - E \left[ Z_2 1_{\{Z_2 < 0\}} \right] = \frac{\bar{v} - i_0 I_0}{\sigma_p} \Phi \left( -\frac{-\bar{v} + i_0 I_0}{\sigma_p} \right) + \phi \left( -\frac{-\bar{v} + i_0 I_0}{\sigma_p} \right).$$

Consequently, we can rewrite the objective of investors as:

$$u_t = \sup_{I_0} \left[ \frac{1}{2} i_0^2 M + E \left[ r_1 \right] k_1 + (\bar{v} - i_0 I_0) \Phi \left( -\frac{-\bar{v} + i_0 I_0}{\sigma_p} \right) 1_{\{I_0 \geq 0\}} \right] + \sigma P \phi \left( -\frac{-\bar{v} + i_0 I_0}{\sigma_p} \right) 1_{\{I_0 \geq 0\}} - Rf I_0 1_{\{I_0 < 0\}},$$

s.t. $k_1 = k_0 + I_0,$

$$i_0 I_0 \leq \left( \bar{v} - \Phi^{-1} \left( 1 - \pi \right) \sigma_p \right) 1_{\{I_0 \geq 0\}}.$$

Notice that investors are risk-neutral if they disinvest, which occurs when $r_1 < Rf$, at which point they will disinvest until $k_1 = 0$. As firms have a Cobb-Douglas production function, the marginal product of capital becomes arbitrarily large, from which follows that $r_1 >> Rf$ a.s. Consequently, investors cannot find it optimal to disinvest in equilibrium. Consequently, we restrict ourselves to the case in which $I_0 \geq 0$.

Let $\lambda_0$ be the Lagrange multiplier on the VaR constraint. The FONC for the program for $I_0$ is then:

$$E \left[ r_1 \right] - \left( \lambda_0 + \Phi \left( \frac{\bar{v} - i_0 I_0}{\sigma_p} \right) \right) i_0 \leq 0 \left( = \right. if \ I_0 > 0 \left. \right).$$

Finally, substituting our results for the truncated normal distribution and the law of motion of capital into the break-even condition of lenders, we arrive at the condition:

$$\frac{\bar{v} \Phi \left( \frac{-\bar{v} + i_0 I_0}{\sigma_p} \right) - \sigma P \phi \left( \frac{-\bar{v} + i_0 I_0}{\sigma_p} \right)}{Rf - \left( 1 - \Phi \left( \frac{-\bar{v} + i_0 I_0}{\sigma_p} \right) \right) i_0},$$

Suppose the VaR constraint binds, then manipulating the break-even condition along with the VaR constraint at equality reveals that:

$$i_0 = \frac{Rf}{1 - \pi + \frac{\pi v - \phi \left( \Phi^{-1} \left( \frac{1 - \pi}{\bar{v} - \Phi^{-1} \left( 1 - \pi \right)} \sigma_p \right) \right)}{v - \Phi^{-1} \left( 1 - \pi \right)}},$$

$$I_0 = \frac{1}{Rf} \left( \bar{v} - \left( \left( 1 - \pi \right) \Phi^{-1} \left( 1 - \pi \right) + \phi \left( \Phi^{-1} \left( 1 - \pi \right) \right) \right) \sigma_p \right).$$

For the comparative statics, it will be useful to recognize that:

$$\phi \left( \Phi^{-1} \left( 1 - \pi \right) \right) - \pi \Phi^{-1} \left( 1 - \pi \right) \geq 0,$$
and therefore:
\[
\frac{dI_0}{d\sigma_P} \leq 0.
\]

Furthermore, it is straightforward to see that:
\[
\frac{di_0}{d\sigma_P} = -\frac{1}{Rf} \frac{1}{\bar{v}} \frac{\bar{v}}{(\bar{v} - \Phi^{-1}(1 - \pi)) \sigma_P^2} \left(\pi \Phi^{-1}(1 - \pi) - \phi(\Phi^{-1}(1 - \pi))\right) \geq 0.
\]

Consequently, from its minimum \(i_0^*\) when \(\sigma_P = 0\), one has that:
\[
i_0 \geq i_0^* = 1 > 0.
\]

Substituting for the optimal level of investment \(I_0\), capital supplied by investors at \(t = 1\) is then:
\[
k_1 = k_0 + \frac{1}{Rf} \left(\bar{v} - ((1 - \pi) \Phi^{-1}(1 - \pi) + \phi(\Phi^{-1}(1 - \pi))) \sigma_P\right).
\]

Imposing market-clearing between the firm and investors at \(t = 1\), or \(K_1 = k_1\), realized output \(Y_2\) at \(t = 2\) is given by:
\[
Y_2 = \frac{Rf k_0 + \bar{v} - ((1 - \pi) \Phi^{-1}(1 - \pi) + \phi(\Phi^{-1}(1 - \pi))) \sigma_P}{Rf} e^{\frac{\bar{v} + \bar{y} + \frac{1 + \alpha}{2} \beta \phi M + \frac{1 + \alpha}{2} \beta^2 \Sigma_{M,\bar{y}} + \sigma_Y^2}} E\left[\frac{1 - \alpha}{w} \frac{\bar{v} + \bar{y} + \frac{1 + \alpha}{2} \beta \phi M + \frac{1 + \alpha}{2} \beta^2 \Sigma_{M,\bar{y}} + \sigma_Y^2}{\bar{v} + \bar{y} + \frac{1 + \alpha}{2} \beta \phi M + \frac{1 + \alpha}{2} \beta^2 \Sigma_{M,\bar{y}} + \sigma_Y^2}\right].
\]

and the rental rate \(r_1\) is:
\[
r_1 = \frac{\alpha}{k_1} E\left[Y_2 | \mathcal{F}_1^M\right] = \alpha \left(\frac{1 - \alpha}{w}\right)^{\frac{1 - \alpha}{\alpha}} e^{\frac{1}{\alpha} \beta \phi M + \frac{1}{2 \alpha} \beta^2 \Sigma_{M,\bar{y}} + \sigma_Y^2}.\]

It then follows that the ex-ante expected output at date 0, is given by:
\[
E[Y_2] = E\left[E\left[Y_2 | \mathcal{F}_1^M\right]\right] = \frac{Rf k_0 + \bar{v} - ((1 - \pi) \Phi^{-1}(1 - \pi) + \phi(\Phi^{-1}(1 - \pi))) \sigma_P}{Rf} e^{\frac{\bar{v} + \bar{y} + \frac{1 + \alpha}{2} \beta \phi M + \frac{1 + \alpha}{2} \beta^2 \Sigma_{M,\bar{y}} + \sigma_Y^2}} E\left[\frac{1 - \alpha}{w} \frac{\bar{v} + \bar{y} + \frac{1 + \alpha}{2} \beta \phi M + \frac{1 + \alpha}{2} \beta^2 \Sigma_{M,\bar{y}} + \sigma_Y^2}{\bar{v} + \bar{y} + \frac{1 + \alpha}{2} \beta \phi M + \frac{1 + \alpha}{2} \beta^2 \Sigma_{M,\bar{y}} + \sigma_Y^2}\right],
\]

where we have applied the Law of Iterated Expectations. It also follows, by the Law of Total Variance, that:
\[
Var[v_2] = Var\left[E\left[v_2 | \mathcal{F}_1^M\right]\right] + E\left[Var\left[v_2 | \mathcal{F}_1^M\right]\right] = Var\left[\hat{v}_1^M\right] + \Sigma_{M,v}.
\]

It follows, since \(Var[v_1] = \sigma_v^2\), that \(Var\left[\hat{v}_1^M\right] = \sigma_v^2 - \Sigma_{M,v}^2\), and we recognize that \(\Sigma_{M,v}^2 \leq \sigma_v^2\).

We then arrive at:
\[
E[Y_2] = \frac{Rf k_0 + \bar{v} - ((1 - \pi) \Phi^{-1}(1 - \pi) + \phi(\Phi^{-1}(1 - \pi))) \sigma_P}{Rf} e^{\frac{\bar{v} + \bar{y} + \frac{1 + \alpha}{2} \beta \phi M + \frac{1 + \alpha}{2} \beta^2 \Sigma_{M,\bar{y}} + \sigma_Y^2 + \frac{1}{2 \sigma_Y^2}}.\]
It is straightforward to see that expected output is decreasing in both the volatility of the asset price, $\sigma_P$, and in the conditional variance of the posterior of $\nu_2$, $\Sigma_{M,vv}$.

Similarly, for the rental rate:

$$E[r_1] = \alpha \left( \frac{1-\alpha}{w} \right)^{\frac{1-\alpha}{\alpha}} e^{\frac{1}{2} \beta (\frac{1}{\alpha} \beta)^2 (\sigma_v^2 - (1-\alpha) \Sigma_{M,vv}) + \frac{1}{2} \sigma_y^2}.$$ 

Finally, we must verify that a corner solution in which the VaR constraint always binds is optimal for investors. The FONC for the investor’s problem reveals that:

$$\lambda_0 = \left( 1 - \pi + \frac{\pi \bar{v} - \phi(1-\pi)\sigma_P}{\bar{v} - \Phi^{-1}(1-\pi)\sigma_P} \right) \frac{1}{\bar{R}} \alpha \left( \frac{1-\alpha}{w} \right)^{\frac{1-\alpha}{\alpha}} e^{\frac{1}{2} \beta (\frac{1}{\alpha} \beta)^2 (\sigma_v^2 - (1-\alpha) \Sigma_{M,vv}) + \frac{1}{2} \sigma_y^2 + \pi - 1},$$

from which it follows that the constraint binds if the multiplier $\lambda_0$ is positive. The first term in parentheses attains its minimum of 1 at $\sigma_P = 0$, and therefore it is sufficient, although not necessary, that:

$$\bar{v} > v^* = -\frac{\alpha}{\beta} \log \left( \alpha \left( \frac{1-\alpha}{w} \right)^{\frac{1-\alpha}{\alpha}} \right) - \frac{\beta}{2\alpha} \left( \sigma_v^2 - (1-\alpha) \Sigma_{M,vv} \right) - \frac{1}{2\beta} \sigma_y^2,$$

for the VaR constraint to bind, and:

$$\sigma_P \leq \sigma_P^* = \frac{\bar{v}}{(1-\pi) \Phi^{-1}(1-\pi) + \phi(1-\pi)},$$

for $I_0$ to be nonnegative.

**Proof of Proposition A3**

To arrive at the beliefs of investors, we first characterize the market beliefs based on only the public information set $\mathcal{F}_t^M$. To derive the market beliefs, we proceed in several steps.

First, we assume the market posterior belief of $(\nu_{t+1}, N_t)$ is jointly Gaussian, $(\nu_{t+1}, N_t) \sim \mathcal{N} \left( (\hat{\nu}_{t+1}^M, \hat{N}_t^M), \Sigma_t^M \right)$, where:

$$\begin{bmatrix} \hat{\nu}_{t+1}^M \\ \hat{N}_t^M \end{bmatrix} = E \begin{bmatrix} u_{t+1} \\ N_t \end{bmatrix} | \mathcal{F}_t^M,$$

$$\Sigma_t^M = \begin{bmatrix} \Sigma_{M,vv}^t & \Sigma_{M,vN}^t \\ \Sigma_{M,Nv}^t & \Sigma_{M,NN}^t \end{bmatrix}.$$ 

Standard results for the Kalman Filter establish that the law of motion of the conditional expectation of the market’s posterior beliefs $(\hat{\nu}_{t+1}^M, \hat{N}_t^M)$ is:

$$\begin{bmatrix} \hat{\nu}_{t+1}^M \\ \hat{N}_t^M \end{bmatrix} = \begin{bmatrix} \rho_v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\nu}_{t+1}^M \\ \hat{\nu}_{t-1}^M \end{bmatrix} + \mathbf{k}_t^M \begin{bmatrix} D_t - \hat{\nu}_t^M \\ \eta_t^H - p_v \hat{\nu}_t^M \end{bmatrix},$$
where:

\[
\mathbf{k}_t^M = Cov \left( \begin{bmatrix} v_{t+1} \\ N_t \end{bmatrix}, \begin{bmatrix} D_t - \hat{v}_t^M \\ \eta_t^H - p_v \rho_v \hat{v}_t^M \end{bmatrix} \mid \mathcal{F}_{t-1}^M \right) \times Var \left( \begin{bmatrix} D_t - \hat{v}_t^M \\ \eta_t^H - p_v \rho_v \hat{v}_t^M \end{bmatrix} \mid \mathcal{F}_{t-1}^M \right)^{-1},
\]

is the Kalman Gain, and the conditional variance \(\Sigma_t^M\) evolves deterministically according to:

\[
\Sigma_t^M = \begin{bmatrix} \rho_v & 0 \\ 0 & 0 \end{bmatrix} \Sigma_{t-1}^M \begin{bmatrix} \rho_v & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_N^2 \end{bmatrix} - \mathbf{k}_t^M Cov \left( \begin{bmatrix} D_t - \hat{v}_t^M \\ \eta_t^H - p_v \rho_v \hat{v}_t^M \end{bmatrix}, \begin{bmatrix} v_{t+1} \\ N_t \end{bmatrix} \mid \mathcal{F}_{t-1}^M \right).
\]

It is straightforward to compute that:

\[
Cov \left( \begin{bmatrix} v_{t+1} \\ N_t \end{bmatrix}, \begin{bmatrix} D_t - \hat{v}_t^M \\ \eta_t^H - p_v \rho_v \hat{v}_t^M \end{bmatrix} \mid \mathcal{F}_{t-1}^M \right) = \begin{bmatrix} \rho_v \Sigma_{t-1}^{M,vv} & \rho_v \sigma_v^2 \\ 0 & p_N \sigma_N^2 \end{bmatrix},
\]

and that:

\[
\Omega_{t-1}^M = Var \left( \begin{bmatrix} D_t - \hat{v}_t^M \\ \eta_t^M - p_v \rho_v \hat{v}_t^M \end{bmatrix} \mid \mathcal{F}_{t-1}^M \right) = \begin{bmatrix} \Sigma_{t-1}^{M,\eta v} + \sigma_D^2 \\ p_v \rho_v \Sigma_{t-1}^{M,\eta v} \end{bmatrix} + \begin{bmatrix} \rho_v \Sigma_{t-1}^{M,vv} + \sigma_v^2 \\ p_v \sigma_v^2 + p_N \sigma_N^2 \end{bmatrix}.
\]

We consider the deterministic steady-state of the Kalman Filter and, consequently, drop all time \(t\) subscripts from conditional variances. We shall verify its existence at the end of the proof.

For \(\eta_t^H \in \mathcal{F}_{t}^M \subseteq \mathcal{F}_t\), I can express \(\eta_t^H\) as:

\[
\eta_t^H = p_v v_{t+1} + p_N N_t = p_v \hat{v}_{t+1} + p_N \hat{N}_t^M,
\]

from which follows that:

\[
p_v \left( v_{t+1} - \hat{v}_{t+1}^M \right) + p_N \left( N_t - \hat{N}_t^M \right) = 0.
\]

As a consequence, it must be that the market beliefs about \(v_t\) and \(N_t\) are ex-post correlated after observing the stock price innovation process \(\eta_t^M\), such that we have the three identities by taking its variance and its covariance with \(v_{t+1} - \hat{v}_{t+1}^M\) and \(N_t - \hat{N}_t^M\):

\[
\Sigma_{M,\eta v} = -\frac{p_v}{p_N} \Sigma_{M,vv},
\]

\[
\Sigma_{M,N} = -\frac{p_v}{p_N} \Sigma_{M,\eta v} = \left( \frac{p_v}{p_N} \right)^2 \Sigma_{M,vv}.
\]
Consequently, as in He and Wang (1995), we need to only compute $\Sigma^{M,vv}$.

Updating the market beliefs to the private beliefs of economic agents can be done in a manner similar to that in He and Wang (1995). Since the market belief acts as a normal prior for investor $i$ who observes the normally distributed private signal $s^i_t$, they update their beliefs by Bayes’ Law in accordance with a linear updating rule. The posterior of investor $i$ is $N(\hat{v}^i_{t+1}, \Sigma^s (i))$, where $\hat{v}^i_{t+1} = E [v^i_{t+1} \mid \mathcal{F}_t^i]$ and $\Sigma^s (i) = E \left[ (v^i_{t+1} - \hat{v}^i_{t+1})^2 \mid \mathcal{F}_t^i \right]$ are given by:

$$\hat{v}^i_{t+1} = \hat{v}^M_{t+1} + \frac{\Sigma^{M,vv}}{\Sigma^{M,vv} + \tau^{-1}_s} (s^i_t - \hat{v}^M_{t+1}),$$

and:

$$\Sigma^s (i)^{-1} = (\Sigma^{M,vv})^{-1} + \tau_s.$$ 

This characterizes the beliefs of investors given the market beliefs.

Since the government does not trade in this benchmark, investors have no incentive to learn about the government’s behavior, and therefore the information acquisition decision is trivial. Given that investors each acquire a private signal $s^i_t$, standard results for CARA utility with normally distributed prices and payoffs establish that the optimal trading policy of investor $i$, $X^i_t$, is given by:

$$X^i_t = \frac{E \left[ D^i_{t+1} + P^i_{t+1} - R^f P^i_t \mid \mathcal{F}_t^i \right]}{\gamma Var \left[ D^i_{t+1} + P^i_{t+1} \mid \mathcal{F}_t^i \right]} \\
= \frac{\left( 1 + p_v (\rho_v - R^f) \right) \left( \hat{v}^i_{t+1} - \hat{v}^M_{t+1} \right) - p_N R^f \hat{N}^i_t}{\left( \begin{array}{c} \hat{v}^i_{t+1} - \hat{v}^M_{t+1} \\ p_v \rho_v \end{array} \right)} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \left( \begin{array}{c} \hat{v}^i_{t+1} - \hat{v}^M_{t+1} \\ p_v \rho_v \end{array} \right)^\prime, \gamma \varphi' \Omega (i) \varphi,$$

where:

$$\varphi = \left( \begin{array}{c} 1 \\ 1 \end{array} \right) + k^M \left( \begin{array}{c} p_v - p_v \\ 0 \end{array} \right),$$

and:

$$\Omega (i) = \Omega^M - \left( \begin{array}{c} 1 \\ p_v \rho_v \end{array} \right) \left( \Sigma^{M,vv} \right)^2 \left( \begin{array}{c} 1 \\ p_v \rho_v \end{array} \right)^\prime,$$

is the conditional variance of $D^i_{t+1}$ and $P^i_{t+1}$ with respect to $\mathcal{F}^i_t$. I can rewrite the above as:

$$X^i_t = \frac{\left( 1 + p_v (\rho_v - R^f) + \left( \begin{array}{c} p_v - p_v \\ 0 \end{array} \right) k^M \left( \begin{array}{c} 1 \\ p_v \rho_v \end{array} \right) \right) \left( \hat{v}^i_{t+1} - \hat{v}^M_{t+1} \right) - p_N R^f \hat{N}^i_t}{\gamma \varphi' \Omega (i) \varphi}.$$ 

Substituting for $\hat{v}^i_{t+1}$, and recognizing from above that:

$$\hat{N}^M_t = N_t + \frac{p_v}{p_N} (v^i_{t+1} - \hat{v}^M_{t+1}),$$

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and therefore that:

\[ \hat{N}_t = \hat{N}_t^M - \frac{p_v}{p_N} (\hat{v}_{t+1}^i - \hat{v}_{t+1}^M) = N_t + \frac{p_v}{p_N} (v_{t+1} - \hat{v}_{t+1}^M) - \frac{p_v}{p_N} (\hat{v}_{t+1}^i - \hat{v}_{t+1}^M), \]

we arrive at:

\[ X_t^i = \left( \varphi' \left[ \begin{array}{c} 1 \\ \frac{1}{p_v \rho_v} \end{array} \right] \frac{\Sigma_{M,vv} \tau_s^{-1}}{\Sigma_{M,vv} + \tau_s^{-1}} (s_t^i - \hat{v}_{t+1}^M) - R^f \left( p_N N_t + p_v (v_{t+1} - \hat{v}_{t+1}^M) \right) \right) \gamma \varphi \Omega (i) \varphi. \]

Aggregating over the demand of investors and imposing market-clearing, we arrive at the two equations for \( p_v \) and \( p_N \):

\[ \varphi' \left[ \begin{array}{c} \frac{1}{p_v \rho_v} \end{array} \right] \frac{\Sigma_{M,vv} \tau_s^{-1}}{\Sigma_{M,vv} + \tau_s^{-1}} = 0, \]

\[ - \frac{R^f p_N}{\gamma \varphi \Omega (i) \varphi} = 1. \]

This completes our characterization of the linear equilibrium.

To see that a stationary equilibrium exists, we can express \( \Sigma^M \) with the Ricatti Equation at its deterministic steady-state:

\[ \Sigma^M = \left[ \begin{array}{cc} \rho_v & 0 \\ 0 & 0 \end{array} \right] + \left[ \begin{array}{cc} \sigma_v^2 & 0 \\ 0 & \sigma_N^2 \end{array} \right] - k^M \left[ \begin{array}{cc} \rho_v \Sigma_{M,vv} & \rho_v (\rho_v \Sigma_{M,vv} + \sigma_v^2) \\ \sigma_v^2 & \rho_N \sigma_N^2 \end{array} \right], \]

where:

\[ k^M = \left[ \begin{array}{cc} \rho_v \Sigma_{M,vv} & \rho_v (\rho_v \Sigma_{M,vv} + \sigma_v^2) \\ \sigma_v^2 & \rho_N \sigma_N^2 \end{array} \right] \left( \Omega^M \right)^{-1}. \]

Consequently, since the conditional variance of beliefs is covariance stationary, it follows that trading strategies are covariance stationary, and consequently are prices, which completes the argument.

**Proof of Proposition A4**

To arrive at the beliefs of investors and the government, we first characterize the market beliefs based on the public information set \( \mathcal{F}_t^M \). To derive the market beliefs, we proceed in several steps. First, we assume the market posterior belief of \((v_{t+1}, N_t, G_{t+1})\) is jointly
Gaussian, \((v_{t+1}, N_t, G_{t+1}) \sim \mathcal{N}\left(\left(\hat{v}_{t+1}, \hat{N}_t, \hat{G}_{t+1}\right), \Sigma_t^M\right)\), where:

\[
\begin{bmatrix}
\hat{v}_{t+1} \\
\hat{N}_t \\
\hat{G}_{t+1}
\end{bmatrix} = E \left[ \begin{bmatrix} v_{t+1} \\ N_t \\ G_{t+1} \end{bmatrix} \mid \mathcal{F}_t \right],
\]

\[
\Sigma_t^M = \begin{bmatrix}
\Sigma_{t,vv} & \Sigma_{t,vN} & \Sigma_{t,vG} \\
\Sigma_{t,Nv} & \Sigma_{t,NN} & \Sigma_{t,NG} \\
\Sigma_{t,Gv} & \Sigma_{t,GN} & \Sigma_{t,GG}
\end{bmatrix}
\]

Standard results for the Kalman Filter then establish that the law of motion of the conditional expectation of the market’s posterior beliefs \(\left(\hat{v}_{t+1}, \hat{N}_t\right)\) is:

\[
\begin{bmatrix}
\hat{v}_{t+1} \\
\hat{N}_{t+1} \\
\hat{G}_{t+1}
\end{bmatrix} = \begin{bmatrix} \rho_v & 0 & 0 \\ 0 & \rho_N & 0 \\ 0 & 0 & \rho_G \end{bmatrix} \begin{bmatrix}
\hat{v}_t \\
\hat{N}_t \\
\hat{G}_t
\end{bmatrix} + \begin{bmatrix}
\Sigma_{t,vv} & \Sigma_{t,vN} & \Sigma_{t,vG} \\
\Sigma_{t,Nv} & \Sigma_{t,NN} & \Sigma_{t,NG} \\
\Sigma_{t,Gv} & \Sigma_{t,GN} & \Sigma_{t,GG}
\end{bmatrix}^{-1} \begin{bmatrix}
D_t - \hat{v}_t \\
\eta_t - p_v \hat{v}_t \\
G_t - G_t_{t|t-1}
\end{bmatrix}
\]

where:

\[
K_t^M = Cov\left[ \begin{bmatrix} v_{t+1} \\ N_t \\ G_{t+1} \end{bmatrix}, \begin{bmatrix} D_t - \hat{v}_t \\ \eta_t - p_v \hat{v}_t \\ G_t - G_t_{t|t-1} \end{bmatrix} \mid \mathcal{F}_t \right]
\]

\[
\times Var\left[ \begin{bmatrix} D_t - \hat{v}_t \\ \eta_t - p_v \hat{v}_t \\ G_t - G_t_{t|t-1} \end{bmatrix} \mid \mathcal{F}_t \right]^{-1}
\]

is the Kalman Gain, and that the conditional variance \(\Sigma_t^M\) evolves deterministically according to:

\[
\Sigma_t^M = \begin{bmatrix} \rho_v & 0 & 0 \\ 0 & \rho_N & 0 \\ 0 & 0 & \rho_G \end{bmatrix} \Sigma_{t-1}^M + \begin{bmatrix} \sigma_v^2 & 0 & 0 \\ 0 & \sigma_N^2 & 0 \\ 0 & 0 & \sigma_G^2 \end{bmatrix}
\]

\[
- K_t^M Cov\left[ \begin{bmatrix} D_t - \hat{v}_t \\ \eta_t - p_v \hat{v}_t \\ G_t - G_t_{t|t-1} \end{bmatrix}, \begin{bmatrix} v_{t+1} \\ N_t \\ G_{t+1} \end{bmatrix} \mid \mathcal{F}_t \right]
\]

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It is straightforward to compute that:

\[
\text{Cov} \left[ \begin{bmatrix}
    v_{t+1} \\
    N_t \\
    G_{t+1} \\
    G_t
\end{bmatrix}, \begin{bmatrix}
    D_t - \hat{v}_t^M \\
    \eta_t^M - p_v \hat{v}_t^M \\
    \eta^M - p_v \hat{v}_G^M \\
    G_t - G_{t|t-1}
\end{bmatrix} \right] \mid \mathcal{F}_{t-1}^M
\]

\[
= \begin{bmatrix}
    p_v \Sigma_{t-1}^{M,v} & p_v \left( \rho_v \Sigma_{t-1}^{M,v} + \sigma_v^2 \right) & p_v \Sigma_{t-1}^{M,vG} \\
    0 & p_N \sigma_N^2 & 0 \\
    0 & p_G \sigma_G^2 & 0 \\
    \Sigma_{t-1}^{M,N} & p_v \rho_v \Sigma_{t-1}^{M,vG} & \Sigma_{t-1}^{M,G_G}
\end{bmatrix},
\]

and that:

\[
\Omega_{t-1}^M = \text{Var} \left[ \begin{bmatrix}
    D_t - \hat{v}_t^M \\
    \eta_t^M - p_v \hat{v}_t^M \\
    \eta^M - p_v \hat{v}_G^M \\
    G_t - G_{t|t-1}
\end{bmatrix} \mid \mathcal{F}_{t-1}^M \right]
\]

\[
= \begin{bmatrix}
    \Sigma_{t-1}^{M,v} + \sigma_D^2 & p_v \rho_v \Sigma_{t-1}^{M,v} & \Sigma_{t-1}^{M,vG} \\
    p_v \rho_v \Sigma_{t-1}^{M,v} & p_v \rho_v \Sigma_{t-1}^{M,v} + \sigma_v^2 & \Sigma_{t-1}^{M,vG} \\
    \Sigma_{t-1}^{M,vG} & p_v \rho_v \Sigma_{t-1}^{M,vG} & \Sigma_{t-1}^{M,G_G}
\end{bmatrix}.
\]

Since \( \Omega_t^M \in \mathcal{F}^M_t \subseteq \mathcal{F}_t \), I can express \( \eta_t^M \) as:

\[
\eta_t^M = \rho_v v_t + p_N N_t = p_v \hat{v}_t^M + p_N \hat{N}_t^M + p_G \hat{G}_t^M,
\]

from which follows that:

\[
p_v \left( v_t - \hat{v}_t^M \right) + p_N \left( N_t - \hat{N}_t^M \right) + p_G \left( G_{t+1} - \hat{G}_{t+1}^M \right) = 0.
\]

As a consequence, it must be that the market beliefs about \( v_t \) and \( N_t \) are ex-post correlated after observing the stock price innovation process \( \eta_t^M \), such that we have the three identities by taking its variance and its covariance with \( v_{t+1} - \hat{v}_{t+1}^M \) and \( N_t - \hat{N}_t^M \):

\[
\begin{align*}
\Sigma_t^{M,vN} &= -\frac{p_v}{p_N} \Sigma_t^{M,v} - \frac{p_G}{p_N} \Sigma_t^{M,vG} \\
\Sigma_t^{M,NN} &= -\frac{p_v}{p_N} \Sigma_t^{M,vN} - \frac{p_G}{p_N} \Sigma_t^{M,NG} \\
\Sigma_t^{M,NG} &= -\frac{p_v}{p_N} \Sigma_t^{M,vG} - \frac{p_G}{p_N} \Sigma_t^{M,G_G}.
\end{align*}
\]

This completes our characterization of the market’s beliefs.

**Proof of Proposition A5**

Updating the market beliefs to each investor’s private beliefs can be done in a manner similar to that in He and Wang (1995). Note that the market beliefs act as the prior for
Proof of Corollary 1

After the system has run for a sufficiently long time, initial conditions will diminish and the conditional variance of the Kalman Filter for the market beliefs $\Sigma^M_t$ will settle down to its deterministic, covariance-stationary steady-state. To see this, let us conjecture that $\Sigma^M_t \to \Sigma^M$. In this proposed steady-state, $\Gamma_t \to \Gamma$, where $\Gamma$ is given by:

$$
\Gamma = \left[ \begin{array}{ccc} 
\sum_{t}^{M,vv} & \sum_{t}^{M,vG_1} \\
\sum_{t}^{M,vN} & \sum_{t}^{M,NG_1} \\
\sum_{t}^{M,vG_1} & \sum_{t}^{M,G_1G_1} 
\end{array} \right] \left[ \begin{array}{ccc} 
\sum_{t}^{M,vv} + (a^i \tau^s)^{-1} & \sum_{t}^{M,vG_1} \\
\sum_{t}^{M,vN} & \sum_{t}^{M,NG_1} \\
\sum_{t}^{M,vG_1} & \sum_{t}^{M,G_1G_1} + [(1 - a^i) \tau^g]^{-1} 
\end{array} \right]^{-1}.
$$
Consequently, since $\Gamma$ is indeed constant, so is $\Sigma_{M,vv}$. Furthermore, the steady-state Kalman Gain $K^M$ is given by:

$$K^M = \begin{bmatrix} \rho_v \Sigma_{M,vv} & \rho_v (\rho_v^2 \Sigma_{M,vv} + \sigma_v^2) & \rho_v \Sigma_{M,vG1} \\ 0 & \rho_N \sigma_N^2 & 0 \\ \Sigma_{M,vG1} & p_v \rho_v \Sigma_{M,vG1} & \Sigma_{M,G1G1} \end{bmatrix} \Omega^{M-1},$$

where:

$$\Omega^M = \begin{bmatrix} \Sigma_{M,vv} + \sigma_D^2 & \rho_v \rho_v \Sigma_{M,vv} & \Sigma_{M,vG1} \\ \rho_v \rho_v \Sigma_{M,vv} & \rho_v \rho_v (\rho_v^2 \Sigma_{M,vv} + \sigma_v^2) + \rho_N^2 \sigma_N^2 + \rho_G^2 \sigma_G^2 & \rho_v \rho_v \Sigma_{M,vG1} \\ \Sigma_{M,vG1} & \rho_v \rho_v \Sigma_{M,vG1} & \Sigma_{M,G1G1} \end{bmatrix}.$$

Consequently, since we have constructed a steady-state for the Kalman Filter for the market beliefs, such a steady-state exists.

**Proof of Proposition A6**

Similar to the problem for the government, it is convenient to define the state vector $\Psi_t = \left[ \hat{\Upsilon}_{t+1}^M, \bar{N}_t^M, \hat{G}_{t+1}^M \right]$ with law of motion:

$$\Psi_{t+1} = \begin{bmatrix} \rho_v 0 0 0 \\ 0 0 0 0 \\ 0 0 0 0 \\ 0 0 1 0 \end{bmatrix} \Psi_t + K^M \varepsilon^M_{t+1},$$

and $\varepsilon^M_{t+1} | \mathcal{F}_t^M \sim N(0_{3x1}, \Omega^M)$ is given by:

$$\varepsilon^M_{t+1} = \begin{bmatrix} D_{t+1} - \hat{\Upsilon}_{t+1}^M \\ \bar{N}_{t+1}^M - p_v \rho_v \hat{\Upsilon}_{t+1}^M \\ \hat{G}_{t+1}^M - \hat{G}_{t+1}^M \end{bmatrix},$$

with $\Omega^M$ given in the proof of Corollary 1.

Given that excess payoffs are normally distributed, we can decompose $R_{t+1}$ as:

$$R_{t+1} = E \left[ R_{t+1} \mid \mathcal{F}_t \right] + \phi^i \varepsilon^S_{t+1}$$

$$= \zeta \Psi_t + \phi^i \zeta \left[ \begin{array}{c} \Sigma_{M,vv} + (a^i \tau_s)^{-1} \\
\Sigma_{M,vG1} \end{array} \right]^{-1} \left[ \begin{array}{c} \Sigma_{M,vG1} \\
\Sigma_{M,vG1} + [(1 - a^i) \tau_g]^{-1} \end{array} \right]^{-1} \left[ \begin{array}{c} \hat{s}_t^i - \hat{s}_{t+1}^M \\
g_t^i - \hat{G}_{t+1}^M \end{array} \right] + \phi^i \varepsilon^S_{t+1}$$

$$= \zeta \Psi_t + \phi^i \zeta \left[ \begin{array}{c} \Sigma_{M,vv} + (a^i \tau_s)^{-1} \\
\Sigma_{M,vG1} \end{array} \right]^{-1} \left[ \begin{array}{c} \Sigma_{M,G1G1} + [(1 - a^i) \tau_g]^{-1} \\
\Sigma_{M,vG1} + (a^i \tau_s)^{-1} \end{array} \right]^{-1} \left[ \begin{array}{c} \hat{s}_t^i - \hat{s}_{t+1}^M \\
g_t^i - \hat{G}_{t+1}^M \end{array} \right] + \phi^i \varepsilon^S_{t+1},$$
where:
\[ \varepsilon_{t+1}^{S_i} = \left[ \begin{array}{c} D_{t+1} - \hat{v}_{t+1}^i \\ \eta_{t+1}^M - p_v \hat{v}_{t+1}^i \\ G_{t+1} - \hat{G}_{t+1}^i \end{array} \right], \]
and:
\[ \varsigma = \left[ \begin{array}{c} 1 + p_v (\rho_v - R^f) \\ -p_N R^f \\ p_g - R^f p_G \\ -R^f p_g \end{array} \right], \]
\[ \phi = \left[ \begin{array}{c} 1 \\ 0 \\ 0 \\ p_G - p_G \\ p_g \end{array} \right] + K^{M_i} \left[ \begin{array}{c} p_v \\ 0 \\ p_G \\ p_g \end{array} \right]. \]

In this decomposition, we have updated the investor’s beliefs sequentially from the market beliefs following Bayes’ Rule as:
\[
E \left[ R_{t+1} \mid \mathcal{F}_i \right] = E \left[ R_{t+1} \mid \mathcal{F}_M \right] + \phi' \omega \left[ \begin{array}{c} \sum^{M,vv} + (a^i \tau_s)^{-1} \\ \sum^{M,vG_1} \\ \sum^{M,G_1G_1} + [(1 - a^i) \tau_g]^{-1} \end{array} \right]^{-1} \left[ \begin{array}{c} s_t^i - \hat{v}_{t+1}^i \\ g_t^i - \hat{G}_{t+1}^i \end{array} \right]
\]
\[
= \varsigma \Psi_i + \phi' \omega \left[ \frac{\sum^{M,G_1G_1} + [(1 - a^i) \tau_g]^{-1} - \sum^{M,vG_1}}{\sum^{M,vv} + (a^i \tau_s)^{-1}} \right] \left[ \frac{\sum^{M,G_1G_1} + [(1 - a^i) \tau_g]^{-1} - \sum^{M,vG_1}}{\sum^{M,vv} + (a^i \tau_s)^{-1}} \right] \left[ \begin{array}{c} s_t^i - \hat{v}_{t+1}^i \\ g_t^i - \hat{G}_{t+1}^i \end{array} \right],
\]
where, as in Proposition 4:
\[ \omega = Cov \left[ \begin{array}{c} \varepsilon_{t+1}^M \\ g_t^i - \hat{G}_{t+1}^i \end{array} \right] \mid \mathcal{F}_M \]
\[ = \left[ \begin{array}{c} \sum^{M,vv} \\ \sum^{M,vG_1} \\ \rho_v \sum^{M,vG_1} \\ \rho_v \sum^{M,vG_1} \\ \rho_v \sum^{M,G_1G_1} \end{array} \right]. \]

Similarly, by Bayes’ Rule, \( \varepsilon_{t+1}^S \mid \mathcal{F}_i \sim N \left( 0_{2 \times 1}, \Omega^S \right) \), where:
\[ \Omega^S = \Omega^M - \omega \left[ \begin{array}{c} \sum^{M,G_1G_1} + [(1 - a^i) \tau_g]^{-1} \\ -\sum^{M,vG_1} \\ \sum^{M,vv} + (a^i \tau_s)^{-1} \end{array} \right] \omega' \]
\[
= \frac{\sum^{M,vv} + (a^i \tau_s)^{-1}}{\sum^{M,vv} + (a^i \tau_s)^{-1}} \left[ \sum^{M,G_1G_1} + [(1 - a^i) \tau_g]^{-1} - (\sum^{M,vG_1})^2 \right].
\]

Standard results establish that the investor’s problem is equivalent to the mean-variance optimization program:
\[
\sup_{X_{t+1}(i)} \left\{ R^f W + X_i E \left[ R_{t+1} \mid \mathcal{F}_i \right] - \frac{\gamma}{2} X_i Var \left[ R_{t+1} \mid \mathcal{F}_i \right] \right\}.
\]

Importantly, since the investors have to form conditional expectations about excess payoffs at \( t + 1 \), they must form conditional expectations about the government’s future trading
Given that the investors are price-takers, from the FOC we see that the optimal investment of investor $i$ in the risky asset is given by:

$$X^i_t = \frac{E[R_{t+1} \mid F^M_t]}{\gamma Var[R_{t+1} \mid F^M_t]} \phi \omega \left[ \frac{\Sigma^M G_1 G_1 + [(1 - \alpha^i) \tau_g]^{-1} - \Sigma^M v G_1}{\Sigma^M v G_1} \right] \left[ s^i_t - \hat{\nu}^{i+1}_t \right] g^M_t - \hat{G}^M_{t+1}$$

This completes our characterization of the optimal trading policy of the investors.

### Proof of Proposition A7

Each investor faces the optimization problem (A1) given in the main paper. It then follows that investor $i$ will choose to learn about the payoff fundamental $v_t$ (i.e., $a^i_t = 1$) with probability $\lambda$:

$$\lambda = \left\{ \begin{array}{ll} 1, & Q < 0 \\ (0, 1), & Q = 0 \\ 0, & Q > 0 \end{array} \right.$$

where:

$$Q = \phi' (M (0) - M (1)) \phi = \phi \omega \left[ \begin{array}{c} -\frac{1}{\Sigma_{M,v} + \tau_s^{-1}} \\ 0 \\ \frac{1}{\Sigma_{M,v} G_1 G_1 + \tau_g} \end{array} \right] \phi'. $$

Given $\omega$, we can expand out this condition to arrive at:

$$Q = \left( \begin{array}{c} (1 + (p_\delta - p_\nu) K_{2,1}^M + (p_\delta - p_\nu) K_{3,1}^M + (p_\delta - p_\nu) K_{3,1}^M G_1) \Sigma_{M,v} G_1 + \tau_g^{-1} \\
+(1 + (p_\delta - p_\nu) K_{2,2}^M + (p_\delta - p_\nu) K_{3,2}^M + (p_\delta - p_\nu) G_1 K_{3,2}^M) \frac{\Sigma_{M,v} + \tau_s^{-1}}{\Sigma_{M,v} G_1 G_1 + \tau_g} \end{array} \right)^2$$

Recognizing that $\phi' \omega = Cov \left[ R_{t+1}, \left[ \begin{array}{c} v_{t+1} \\ G_{t+1} \end{array} \right] \mid F^M_t \right]$, we can rewrite the above more generally as:

$$Q = \frac{Cov \left[ R_{t+1}, G_{t+1} \mid F^M_t \right]^2}{\Sigma_{M,G_1 G_1} + \tau_g^{-1}} - \frac{Cov \left[ R_{t+1}, v_{t+1} \mid F^M_t \right]^2}{\Sigma_{M,v} + \tau_s^{-1}}.$$
Proof of Proposition A8

In the special case that $\rho_v = 0$, it follows that the Kalman Gain, the steady-state market beliefs, and the $Q$–statistic for information acquisition satisfy:

$$
\mathbf{K}^M = \begin{bmatrix}
0 & \frac{p_v \sigma^2_v}{p_v \sigma^2_v + p_N \sigma^2_N + p_G \sigma^2_G} & 0 \\
\frac{p_v \sigma^2_v}{p_v \sigma^2_v + p_N \sigma^2_N + p_G \sigma^2_G} & 0 & 0 \\
\frac{p_v \sigma^2_v}{p_v \sigma^2_v + p_N \sigma^2_N + p_G \sigma^2_G} & 0 & 0 \\
0 & 0 & 1
\end{bmatrix},
$$

and:

$$
\Sigma^M = \begin{bmatrix}
\frac{p_v \sigma^2_v + p_G \sigma^2_G + \frac{1}{p_v} p_v \sigma^2_v \sigma^2_N}{p_v \sigma^2_v + p_N \sigma^2_N + p_G \sigma^2_G} p_g & -\frac{p_v \sigma^2_v p_N \sigma^2_N}{p_v \sigma^2_v + p_N \sigma^2_N + p_G \sigma^2_G} & -\frac{p_v \sigma^2_v p_G \sigma^2_G}{p_v \sigma^2_v + p_N \sigma^2_N + p_G \sigma^2_G} \\
-\frac{p_v \sigma^2_v p_N \sigma^2_N}{p_v \sigma^2_v + p_N \sigma^2_N + p_G \sigma^2_G} & \frac{p_v \sigma^2_v \sigma^2_N}{p_v \sigma^2_v + p_N \sigma^2_N + p_G \sigma^2_G} \tau^{-1} & 0 \\
-\frac{p_v \sigma^2_v p_G \sigma^2_G}{p_v \sigma^2_v + p_N \sigma^2_N + p_G \sigma^2_G} & 0 & \frac{p_v \sigma^2_v \sigma^2_G}{p_v \sigma^2_v + p_N \sigma^2_N + p_G \sigma^2_G} \tau^{-1}
\end{bmatrix},
$$

and:

$$
Q = \left( \frac{p_N \sigma^2_N + p_G \sigma^2_G + \frac{1}{p_v} p_v \sigma^2_v - p_v p_G \sigma^2_G}{p_v \sigma^2_v + p_N \sigma^2_N + p_G \sigma^2_G} p_g \left( p_v^2 + p_N^2 \sigma^2_N \right) - p_v p_G \right)^2 \left( \frac{\sigma^2_v \sigma^2_G}{p_v^2 \sigma^2_v + p_N^2 \sigma^2_N + p_G^2 \sigma^2_G} \right)^2
$$

$$
\left[ p^2 \sigma^2_N + p_G^2 \sigma^2_G - \frac{p_N \sigma^2_N + p_G \sigma^2_G + \frac{1}{p_v} p_v \sigma^2_v - p_v p_G \sigma^2_G}{p_v \sigma^2_v + p_N \sigma^2_N + p_G \sigma^2_G} p_g p_v p_G \right]^2 \left( \frac{\sigma^2_v \sigma^2_G}{p_v^2 \sigma^2_v + p_N^2 \sigma^2_N + p_G^2 \sigma^2_G} \right)^2,
$$

respectively.

In a government-centric equilibrium, $p_v = 0$, and, from the market-clearing conditions, $p_g$ and $p_G$ satisfy:

$$
p_g = -\frac{p_N \sigma^2_N}{1 - \vartheta_N} \sqrt{\frac{p_N^2 \sigma^2_N}{p_N^2 \sigma^2_N + p_G^2 \sigma^2_G} \vartheta^2_N},
$$

$$
p_G = \frac{1}{Rf} (1 - \vartheta_N) \frac{p_N p_G^2 \sigma^2_N}{p_N^2 \sigma^2_N + p_G^2 \sigma^2_G} \sigma^2_G,
$$

from which follows that $p_G$ is given by $p_G^2 = x p_N^2 \sigma^2_N$ where $x$ satisfies:

$$
x (1 + x)^3 = \left( \frac{\vartheta_N}{Rf} \sigma^2_G \right)^2,
$$
where \( x \) is increasing in \( \frac{\vartheta}{R^f \sigma_G} \). It then follows that \( Q \) reduces to:

\[
Q = \frac{(\sigma_G^2 - R f \frac{x}{1 - \vartheta_N})^2}{\sigma_G^2 + (1 + x) \tau_g^{-1}} \left( \frac{\vartheta_N}{1 - \vartheta_N} \right)^2 \frac{p_N^2}{\sigma_v^2 + \sigma_\vartheta^2} \frac{\vartheta_N}{\sigma_v^2 + \tau_\vartheta^{-1}} (1 + x)^2 \left( \frac{\sigma_G^2}{1 + x} \right)^2,
\]

which suggests that, for \( Q \geq 0 \), it must be the case that:

\[
p_N^2 > \bar{p}_N^2 = \frac{\sigma_v^2 \sigma_G^2 + (1 + x) \tau_g^{-1} 1 - \vartheta_N}{\sigma_v^2 + \tau_\vartheta^{-1}} \left( \frac{1 - \vartheta_N}{\vartheta_N} \right)^2 \left( \frac{1 + x}{\sigma_v^2 - R f \frac{x}{1 - \vartheta_N}} \right)^2.
\]

Furthermore, it is straightforward to compute that:

\[
\phi \Omega^M \phi = \sigma_v^2 + \sigma_D^2 + \sigma_G^2 \left( \frac{1}{1 + x} - \frac{\vartheta_N}{1 - \vartheta_N} \right)^2 p_N^2 \sigma_G^2 + \left( 1 + \frac{1 + x \frac{1}{1 - \vartheta_N}}{1 + x} \frac{\sigma_G^2}{\sigma_v^2 + \sigma_D^2} \right)^2 p_N^2 \sigma_G^2,
\]

and therefore, from market-clearing, that \( p_N \) also satisfies:

\[
0 = \left( \sigma_G^2 \frac{\sigma_v^2 + 2 (1 + x) \tau_g^{-1}}{\sigma_G^2 + (1 + x) \tau_g^{-1} \frac{\vartheta_N}{1 + x} \frac{1}{1 - \vartheta_N}} \left( \frac{1 - \vartheta_N}{\vartheta_N} \right)^2 + \frac{1 + \frac{1 + x \frac{1}{1 - \vartheta_N}}{1 + x} \frac{\sigma_G^2}{\sigma_v^2 + \sigma_D^2}}{1 + x} \right) \frac{\sigma_G^2}{p_N^2}
\]

\[
+ \frac{R f}{1 - \vartheta_N} \frac{1 + x}{\sigma_G^2 + (1 + x) \tau_g^{-1}} p_N + \sigma_v^2 + \sigma_D^2.
\]

It follows that \( p_N \) is given by the two roots of the above quadratic form:

\[
p_N = -\frac{1}{2 \sigma_v^2 c} \frac{R f}{1 - \vartheta_N} \pm \sqrt{\frac{1}{2 \sigma_v^2 c} - \frac{\sigma_G^2 + \sigma_D^2}{\sigma_G^2 c}}
\]

where:

\[
c = \sigma_G^2 \frac{\sigma_v^2 + 2 (1 + x) \tau_g^{-1}}{\sigma_G^2 + (1 + x) \tau_g^{-1} \frac{\vartheta_N}{1 + x} \frac{1}{1 - \vartheta_N}} \left( \frac{1 - \vartheta_N}{\vartheta_N} \right)^2 + \frac{1 + \frac{1 + x \frac{1}{1 - \vartheta_N}}{1 + x} \frac{\sigma_G^2}{\sigma_v^2 + \sigma_D^2}}{1 + x} \geq 0,
\]

and \( c = c(\vartheta_N, R f, \sigma_G) \). When \( y \) exists, one consequently has that \( y < 0 \). Selecting the less negative root, and recognizing that \( Q \geq 0 \) whenever \( -y \geq \bar{y} \), we can express this condition as:

\[
\frac{\sqrt{\sigma_v^2 + \tau_\vartheta^{-1}}}{\sigma_v^2} \left( \frac{1}{2 \sigma_v c} \frac{R f}{1 - \vartheta_N} - \sqrt{\frac{1}{2 \sigma_v c} - \frac{R f}{\sigma_G^2 c}} - \frac{\sigma_v^2 + \sigma_D^2}{\sigma_G^2 c} \right) \geq (1 + x) \sqrt{\sigma_G^2 + (1 + x) \tau_g^{-1} \left( \frac{1 - \vartheta_N}{\sigma_G^2 - R f \frac{x}{1 - \vartheta_N}} \right)^2}.
\]
Notice that the LHS of equation (1) is always nonnegative, since it is \(-\frac{\sqrt{\sigma_2^2 + \tau_s^2}}{\sigma_e^2} p_N \sigma_N\), and that \(c\) and the RHS of equation (1) is independent of \(\{\sigma_N, \sigma_u, \sigma_D\}\) since \(x = x (\vartheta_N, R^f, \sigma_G)\).

Since it is straightforward to compute that:

\[
\frac{dp_N \sigma_N}{d\sigma_N} = \frac{1}{\sigma_N 2\sigma_N c 1 - \vartheta_N} \frac{R^f}{\sqrt{\left(\frac{1}{2\sigma_N c 1 - \vartheta_N}\right)^2 - \frac{\sigma_2^2 + \sigma_2^2}{c}}} < 0,
\]

\[
\frac{dp_N \sigma_N}{d\sigma_D} = -\frac{\sigma_D}{c} \frac{R^f}{\sqrt{\left(\frac{1}{2\sigma_N c 1 - \vartheta_N}\right)^2 - \frac{\sigma_2^2 + \sigma_2^2}{c}}} < 0,
\]

it follows that, since \(y = -p_N \sigma_N\), the LHS is increasing in \(\sigma_N\) and \(\sigma_D\). Consequently, the existence condition for a government-centric equilibrium relaxes as \(\sigma_N\) and \(\sigma_D\) increase, and therefore a government-centric equilibrium is more likely to exist the higher are \(\sigma_N\) and \(\sigma_D\).

Finally, with respect to \(\sigma_u\), we recognize that, as \(\sigma_u \to 0\), \(-\frac{\sqrt{\sigma_2^2 + \tau_s^2}}{\sigma_e^2} p_N \sigma_N \to \infty\), and consequently the LHS exceeds the RHS and \(Q > 0\). Since \(-\frac{\sqrt{\sigma_2^2 + \tau_s^2}}{\sigma_e^2} p_N \sigma_N\) is continuous in \(v\), it follows that a government-centric equilibrium exists within a neighborhood of \(\sigma_u = 0\), and consequently exists for \(\sigma_u\) sufficiently small.

**Proof of Proposition A9**

Since the government does not have any additional information to that of the market, it has the market beliefs. As described in the main paper, it is convenient to define the state vector \(\Psi_t\), which follows a VAR(1) process in the covariance-stationary equilibrium of the economy given by Proposition A5.

Given the results in Proposition A5, the government’s policy rule:

\[
X_t^G = \vartheta_N \tilde{N}_t^M + \sqrt{\text{Var} \left[ \vartheta_N \tilde{N}_t^M \mid F_{t-1}, \{a_{ti}^j\}_i \right]} G_t,
\]

and it follows that the conditional price volatility can be expressed as:

\[
\text{Var} \left[ \Delta P_{t+1} \mid F_t^M \right] = \left( \phi - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)' \Omega^M \left( \phi - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right),
\]

where \(\vartheta = \begin{bmatrix} 0 & \vartheta_N & 0 & 0 \end{bmatrix}'\).
Finally, we can express the conditional uncertainty about the deviation in the asset price from its fundamentals as:

\[
F = \text{Var} \left[ P_{t+1} - p_v v_{t+2} \mid \mathcal{F}_t^M \right]
\]

\[
= \text{Var} \left[ \left( \phi - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)' \varepsilon_{i+1} - p_v \left( v_{t+2} - \rho_v v_{i+1} \right) \mid \mathcal{F}_t^M \right]
\]

\[
= \left( \phi - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)' \Omega^M \left( \phi - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + p_v^2 \left( \rho_v^2 \Sigma^{M,\nu \nu} + \sigma_v^2 \right)
\]

\[
-2p_v \left( \phi - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)' \rho_v \left( \rho_v^2 \Sigma^{M,\nu \nu} + \sigma_v^2 \right) \rho_v \Sigma^{M,\nu G_1}.
\]

It follows in the covariance-stationary equilibrium that we can express the government’s objective as:

\[
U^G = \sup_{\phi} -\gamma_{\phi} \left( \phi - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)' \Omega^M \left( \phi - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) - \gamma_v F.
\]

References